Intermediate public economics 8
(Im)possibility theorems

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Aggregation rule

Preference

- Let $\mathcal{C} \subseteq \mathcal{R}$ the set of all complete preferences on X
- Let $\mathcal{B} \subseteq \mathcal{C}$ be the set of all transitive preferences in $\mathcal{C}$
- Consider an economy consisting of $n \geq 2$ individuals and put $I := \{1, 2, \dots, n\}$
- Preference profile is an n-tuple $\rho := (\succsim_1, \dots, \succsim_n) \in \mathcal{D}^n$ for some $\mathcal{D} \subseteq \mathcal{R}$

Preference aggregation

- An aggregation rule on $\mathcal{D}^n$ is a function $\succsim: \mathcal{D}^n \to \mathcal{C}$, which assigns an aggregate preference on X to every possible preference profile in $\mathcal{D}^n$
- Denote by $\succsim_\rho \in \mathcal{C}$ the aggregate preference associated with a preference profile $\rho \in \mathcal{D}^n$ under an aggregation rule $\succsim: \mathcal{D}^n \to \mathcal{C}$

Axioms

Transitivity, unanimity, and independence

- An aggregation rule $\succsim: \mathcal{D}^n \to \mathcal{C}$ is said to be
  - transitive if $\succsim_\rho \in \mathcal{B}$ for every $\rho \in \mathcal{D}^n$
  - unanimous if

$$x \succ_i x' \text{ for all } i \in I \implies x \succ_\rho x' \tag{1}$$

    for every $\rho = (\succsim_1, \dots, \succsim_n) \in \mathcal{D}^n$ and for every $x, x' \in X$.
  - independent if

$$\succsim_i = \succsim_i' \text{ on } \{x, x'\} \; \forall i \in I \implies \succsim_\rho = \succsim_{\rho'} \text{ on } \{x, x'\} \tag{2}$$

    for every $\rho = (\succsim_1, \dots, \succsim_n), \rho' = (\succsim_1', \dots, \succsim_n') \in \mathcal{D}^n$ and for every $x, x' \in X$
  - The last axiom is also called the independence of irrelevant alternatives

Axioms (cont'd)

Dictatorship and non-dictatorship

- An aggregation rule $\succsim: \mathcal{D}^n \to \mathcal{C}$ is said to be
  - dictatorial if there exists $i \in I$ such that
    $$x \succ_i x' \implies x \succ_\rho x' \tag{3}$$
    for every $\rho = (\succsim_1, \ldots, \succsim_n) \in \mathcal{D}^n$ and for every $x, x' \in X$
  - non-dictatorial if it is not dictatorial

Decisive group

- A subset $L \subseteq I$ is said to be decisive under $\succsim$ if
  $$x \succ_i x' \text{ for all } i \in L \implies x \succ_\rho x' \tag{4}$$
  for every $\rho = (\succsim_1, \ldots, \succsim_n) \in \mathcal{D}^n$ and for every $x, x' \in X$
- Unanimous if $L = I$ and dictatorial if $L = \{i\}$

Arrow's impossibility theorem

Theorem

- If an aggregation rule on $\mathcal{D}^n = \mathcal{B}^n$ is transitive, unanimous, and independent, then it must be dictatorial
- In other words, there is no transitive aggregation rule defined on $\mathcal{B}^n$ which simultaneously satisfies unanimity, independence, and non-dictatorship

Proof

- A couple of elementary proofs are available
- Perhaps the simplest is the one which uses the field expansion lemma
- Assume a transitive aggregation rule $\succsim$ on $\mathcal{B}^n$ is unanimous and independent and find a subset $L \subseteq I$ with $|L| = 1$ such that L is decisive under $\succsim$
- See the lecture note for more details

Implications of Arrow's theorem

General impossibility

- Arrow suggested that any generally acceptable aggregation rule must satisfy unanimity, independence, and non-dictatorship
- But he showed that every aggregation rule inevitably violates at least one of the three conditions
- Hence, designing a 'generally acceptable aggregation rule' is impossible

Is it possible to escape from the impossibility?

- There are several ways of escaping the impossibility
- Among the most promising are:
  - giving up the independence axiom
  - restricting the domain $\mathcal{D}^n$ to a proper subset of $\mathcal{B}^n$

On independence axiom

Normative implication

- Intensity of preference is all deemed irrelevant
- When two alternatives are compared, their relative positions against the other alternatives are not taken into account (is this always reasonable?)
- Borda rule violates independence (See Table), but one could argue that this is reasonable

#	60	40
1st	x_1	x_2
2nd	x_2	x_1
3rd	x_3	x_3

$\rightarrow$

#	60	40
1st	x_1	x_2
2nd	x_2	x_3
3rd	x_3	x_1

Majoritarian rule revisited

Majoritarian aggregation rule

- For each $\rho \in \mathcal{D}^n$, define $\#_\rho : X \times X \to \mathbb{Z}_+$ by

$$\#_\rho(x, x') := |\{i \in I \,|\, x \succ_i x'\}| \quad \forall x, x' \in X \tag{5}$$

- An aggregation rule $\succsim: \mathcal{D}^n \to \mathcal{B}$ defined by

$$x \succsim_\rho x' \iff \#_\rho(x, x') \geq n/2 \tag{6}$$

  is called the majoritarian rule

Arrow's theorem and Condorcet paradox

- Easy to see that the majoritarian rule satisfies unanimity, independence, and non-dictatorship
- If $\mathcal{D}^n = \mathcal{B}^n$, then transitivity must be violated by Arrow's theorem (cf. Condorcet paradox)
- What if we restrict the domain $\mathcal{D}^n$?

Single-peaked preference

Single-peakedness

- Assume $X \subseteq \mathbb{R}$
- A preference $\succsim_i$ on X is said to be single-peaked if there exists $x_i^* \in X$ such that

$$x < x' < x_i^* \text{ or } x_i^* < x' < x \implies x_i^* \succ_i x' \succ_i x \qquad (7)$$

- Here x_i^* is a kind of 'bliss point' of individual i

Majoritarian rule on single-peaked preferences

- Denote by $\mathcal{A} \subseteq \mathcal{B}$ the set of all single-peaked, complete, and transitive preferences
- If we restrict $\mathcal{D}^n = \mathcal{A}^n$, then the majoritarian rule satisfies transitivity as well (median voter theorem)
- Reasonable to restrict the domain this way if X consists of some quantities or political spectrum

Black's median voter theorem

Median voter

- Assume n is odd and restrict the domain as $\mathcal{D}^n = \mathcal{A}^n$
- For each $\rho \in \mathcal{D}^n$, define a permutation $\iota_\rho : I \to I$ by

$$\iota_\rho(i) \leq \iota_\rho(j) \implies x^*_{\iota_\rho(i)} \leq x^*_{\iota_\rho(j)} \tag{8}$$

- Individual $m_\rho \in I$ is called the median voter in a preference profile $\rho \in \mathcal{D}^n$ if $\iota_\rho(m_\rho) = (n-1)/2 + 1$

Theorem

- If $\mathcal{D}^n = \mathcal{A}^n$, the majoritarian rule $\succsim : \mathcal{D}^n \to \mathcal{C}$ is transitive, unanimous, independent, and non-dictatorial
- Moreover, for every $\rho \in \mathcal{D}^n$,

$$x^*_{m_\rho} \succsim_\rho x \text{ for all } x \in X, \tag{9}$$

where $x^*_{m_\rho}$ is the bliss point of the median voter in ρ

Black's median voter theorem (cont'd)

Proof

- For any $x < x^*_{m_\rho}$

$$x_i^* \leq x \implies x \succ_i x^*_{m_\rho}, \tag{10}$$

$$x^*_{m_\rho} \leq x_i^* \implies x^*_{m_\rho} \succ_i x \tag{11}$$

which implies

$$\#_\rho(x, x^*_{m_\rho}) \leq \frac{(n-1)}{2} < \frac{n-1}{2} + 1 \leq \#_\rho(x^*_{m_\rho}, x) \tag{12}$$

and therefore $x^*_{m_\rho} \succsim_\rho x$

- The argument for the case with $x > x^*_{m_\rho}$ is identical
- Transitivity follows from a similar argument
- It should be easy to see that $x^*_{m_\rho}$ is in fact a Condorcet winner

Implications

Party politics

- Consider a two-party system, where two major parties dominate the politics
- Each party chooses a political position (candidate or policy) from $X \subseteq \mathbb{R}$
- Majoritarian voting rule is assumed
- What would be the best strategy for these parties?

Aiming at the median voter's bliss point

- When individuals have single-peaked preferences, their best strategy is to 'make their position as close as the median voter's bliss point'
- Suggesting that political positions of two parties will converge to a moderate one if preferences of individuals are single-peaked

Social choice rule

Social choice rules

- Denote by $\mathcal{S}$ the collection of all nonempty subsets of X
- A social choice rule on $\mathcal{D}^n$ is a function

$$f : \mathcal{D}^n \times \mathcal{S} \to X \tag{13}$$

  such that $f(\rho, S) \in S$ for all $(\rho, S) \in \mathcal{D}^n \times \mathcal{S}$

Notations

- We write $f(\rho) := f(\rho, X)$, i.e., we suppress S in f when $S = X$
- For each $\rho = (\succsim_1, \ldots, \succsim_n) \in \mathcal{D}^n$, we write

$$\succsim_{-i} := (\succsim_1, \ldots, \succsim_{i-1}, \succsim_{i+1}, \ldots, \succsim_n) \tag{14}$$

  so that we may write

$$f(\rho, S) = f(\succsim_i, \succsim_{i-1}, S) \tag{15}$$

Axioms

Transitivity, unanimity, and independence

- A social choice rule $f : \mathcal{D}^n \times \mathcal{S} \to X$ is said to be
  - transitive (or rational) if
    $$x = f(\rho, S_1) \text{ and } x' = f(\rho, S_2) \in S_1 \implies x \notin S_2 \quad (16)$$
    for every $\rho \in \mathcal{D}^n$ and $S_1, S_2 \in \mathcal{S}$
  - unanimous (or weakly Paretian) if
    $$x \succ_i x' \text{ for all } i \in I \implies f(\rho, \{x, x'\}) = x \quad (17)$$
    for every $\rho = (\succsim_1, \ldots, \succsim_n) \in \mathcal{D}^n$ and for every $x, x' \in X$.
  - independent if
    $$\succsim_i = \succsim_i' \text{ on } \{x, x'\} \ \forall i \in I \implies f(\rho, \{x, x'\}) = f(\rho', \{x, x'\})$$
    for every $\rho = (\succsim_1, \ldots, \succsim_n), \rho' = (\succsim_1', \ldots, \succsim_n') \in \mathcal{D}^n$ and for every $x, x' \in X$

Axioms (cont'd)

Dictatorship and strategy proofness

- A social choice rule $f : \mathcal{D}^n \times \mathcal{S} \to X$ is said to be
  – dictatorial if there exists $i \in I$ such that

  $$x \succ_i x' \implies f(\rho, \{x, x'\}) = x \tag{18}$$

  for every $\rho = (\succsim_1, \ldots, \succsim_n) \in \mathcal{D}^n$ and for every $x, x' \in X$

  – non-dictatorial if it is not dictatorial

  – strategy proof if

  $$f(\succsim_i, \succsim_{-i}, S) \succsim_i f(\succsim_i', \succsim_{-i}, S) \quad \forall \succsim_i' \in \mathcal{D} \tag{19}$$

  for every $\rho = (\succsim_1, \ldots, \succsim_n) \in \mathcal{D}^n$ and $S \in \mathcal{S}$
- Notice that strategy proofness requires that truth-telling be a dominant strategy for every player

Theorems

Arrow's impossibility theorem (choice rule version)

- The following is a translation of the Arrow's theorem into the language of social choice rule
- If a social choice rule on $\mathcal{D}^n = \mathcal{B}^n$ is transitive, unanimous, and independent, then it must be dictatorial

Gibbard-Satterthwaite theorem

- Let $\mathcal{P} \subseteq \mathcal{R}$ be the set of all preferences on X such that no two distinct alternatives in X are indifferent
- If a social choice rule on $\mathcal{D}^n = \mathcal{P}^n$ is unanimous and strategy proof, then it must be dictatorial
- Observe the close relationship between independence and strategy-proofness

$\rightarrow$ Restricting the domain opens up the possibility to design a strategy-proof social choice rule

Median voter theorem revisited

Median rule (Majoritarian rule)

- When $X \subseteq \mathbb{R}$ and if we restrict the domain as $\mathcal{D}^n = \mathcal{A}^n$, we can define a social choice rule on $\mathcal{D}^n$ which corresponds to the majoritarian aggregation rule
- A median rule is a social choice rule f defined by

$$f(\rho, S) := x^*_{m_\rho} \quad \forall (\rho, S) \in \mathcal{A}^n \times \mathcal{S}, \tag{20}$$

where $m_\rho \in I$ is the median voter in ρ given S

Median voter theorem (choice rule version)

- Median rule is transitive, unanimous, independent, and non-dictatorial
- Moreover, median rule is strategy proof!
- Median rule always chooses the Condorcet winner

Implications

Single-peakedness matters

- We can design a strategy-proof, unanimous, non-dictatorial social choice rule as long as individuals' preferences are all single-peaked
- Reasonable to assume single-peakedness when the issue dimension is clear

Deliberation might help

- Deliberation is a process of thoughtfully weighing options, emphasizing the use of logic and reason
- Shaping individuals' preferences before voting, which possibly makes them approximately single-peaked
- This can be done by facilitating the formation of 'meta-agreement,' an agreement on what is the most important issue dimension