# Intermediate public economics 7
## Voting rules

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# Individual and social choice

**Individual choice**

- Alternatives: $S := \{x_1, x_2, x_3, x_4, x_5\}$
- If an individual has a preference $\succsim$ on $S$ such that

$$x_1 \succ x_4 \succ x_5 \succ x_3 \succ x_2, \tag{1}$$

  then she would certainly choose $x_1$
- If another individual has a preference $\succsim'$ on $S$ such that

$$x_2 \succ' x_5 \succ' x_4 \succ' x_3 \succ' x_1, \tag{2}$$

  then he would certainly choose $x_2$

**Social choice (or collective choice)**

- But what if this choice concerns both of them?
- How can they collectively make a decision?
- How should we aggregate their preferences?

# Preference aggregation

## Notations

- Let $X$ be the set of all possible alternatives
- Denote by $\mathcal{S}$ the collection of all nonempty subsets of $X$
- Denote by $\mathcal{R}$ the set of all possible preferences on $X$

## Social choice rules

- A social choice rule is a function

$$f : \mathcal{R}^n \times \mathcal{S} \to X \tag{3}$$

  such that $f(\rho, S) \in S$ for all $(\rho, S) \in \mathcal{R}^n \times \mathcal{S}$
- Procedure:
  1. specify the subset $S \in \mathcal{S}$ of actual alternatives
  2. ask individuals to reveal their preference $\succsim_i \in \mathcal{R}$ on $S$ so that we have data $\rho := (\succsim_1, \succsim_2, \ldots, \succsim_n) \in \mathcal{R}^n$
  3. we then collectively choose $f(\rho, S) \in S$

# Which alternative should we choose?

**Preferences of 55 individuals over 5 alternatives**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Plurality rule chooses** $x_1$

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Run-off rule first chooses $x_1$ and $x_2$**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Run-off rule chooses $x_2$ eventually**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Sequential-elimination rule first eliminates** $x_5$

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Sequential-elimination rule then eliminates $x_4$**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Sequential-elimination rule then eliminates $x_2$**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Sequential-elimination rule eventually chooses $x_3$**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Borda rule (simple scoring rule) chooses** $x_4$

| point | # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|---|
| 4 | 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 3 | 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 2 | 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 1 | 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 0 | 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Pairwise-majority rule prefers $x_5$ to $x_1$**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Pairwise-majority rule prefers $x_5$ to $x_2$**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Pairwise-majority rule prefers $x_5$ to $x_3$**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Pairwise-majority rule prefers $x_5$ to $x_4$**

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Which alternative should we choose?

**Pairwise-majority rule therefore chooses** $x_5$

| # | 18 | 12 | 10 | 9 | 4 | 2 |
|---|---|---|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_5$ |
| 2nd | $x_4$ | $x_5$ | $x_2$ | $x_3$ | $x_2$ | $x_3$ |
| 3rd | $x_5$ | $x_4$ | $x_5$ | $x_5$ | $x_4$ | $x_4$ |
| 4th | $x_3$ | $x_3$ | $x_4$ | $x_2$ | $x_3$ | $x_2$ |
| 5th | $x_2$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ | $x_1$ |

example 1: taken from Malkevitch (1990)

# Implications

**Voting rule matters**

- Result of social choice (i.e., voting) completely depends on how we design the voting rule
- Simply because a candidate wins an election does not mean the result 'correctly' reflects the public opinion
- Should be very careful in interpreting election results

**Choosing how to choose**

- Then how should we design voting rules?
- Good voting rule $f : \mathcal{R}^n \times \mathcal{S} \to X$ should:
  - not generate an unreasonable output for any possible input $(\rho, S) \in \mathcal{R}^n \times \mathcal{S}$
  - not be susceptible to strategic manipulations
- Let us examine these points in more detail

# Criteria of good voting rules

**Commonly-used criteria**

- Almost always required:
  - unanimity (weak Pareto)
  - non-dictatorship
  - weak axiom of revealed preference (WARP)
- Widely accepted, but somewhat controversial:
  - Condorcet consistency
  - binary independence
- Criteria against strategic manipulations:
  - participation criterion
  - preference revealing
- In designing a voting rule, we need to decide which criteria we want it to satisfy

# Unanimity

**Unanimity criterion**

- Consider the following profile of preferences
- Note that $x_2$ is preferred over $x_4$ unanimously
- Any good voting rule should not choose $x_4$

| # | 1 | 1 | 1 |
|---|---|---|---|
| 1st | $x_2$ | $x_1$ | $x_3$ |
| 2nd | $x_4$ | $x_2$ | $x_1$ |
| 3rd | $x_3$ | $x_4$ | $x_2$ |
| 4th | $x_1$ | $x_3$ | $x_4$ |

# Unanimity (cont'd)

**Tournament rule**

- Consider a tournament rule which successively compares alternatives $x_1 \to x_2 \to x_3 \to x_4$
- This rule chooses $x_4$, violating the unanimity criterion

| # | 1 | 1 | 1 |
|---|---|---|---|
| 1st | $x_2$ | $x_1$ | $x_3$ |
| 2nd | $x_4$ | $x_2$ | $x_1$ |
| 3rd | $x_3$ | $x_4$ | $x_2$ |
| 4th | $x_1$ | $x_3$ | $x_4$ |

# Unanimity (cont'd)

**Tournament rule**

- Consider a tournament rule which successively compares alternatives $x_1 \to x_2 \to x_3 \to x_4$
- This rule chooses $x_4$, violating the unanimity criterion

| # | 1 | 1 | 1 |
|---|---|---|---|
| 1st | $x_2$ | $x_1$ | $x_3$ |
| 2nd | $x_4$ | $x_2$ | $x_1$ |
| 3rd | $x_3$ | $x_4$ | $x_2$ |
| 4th | $x_1$ | $x_3$ | $x_4$ |

# Unanimity (cont'd)

**Tournament rule**

- Consider a tournament rule which successively compares alternatives $x_1 \to x_2 \to x_3 \to x_4$
- This rule chooses $x_4$, violating the unanimity criterion

| # | 1 | 1 | 1 |
|---|---|---|---|
| 1st | $x_2$ | $x_1$ | $x_3$ |
| 2nd | $x_4$ | $x_2$ | $x_1$ |
| 3rd | $x_3$ | $x_4$ | $x_2$ |
| 4th | $x_1$ | $x_3$ | $x_4$ |

# Weak axiom of revealed preference

**WARP**

- Outcome should not be reversed even if another alternative is added:

$$f(\rho, \{x_1, x_2\}) = x_1 \implies f(\rho, \{x_1, x_2, x_3\}) \neq x_2 \tag{4}$$

- Intended to rule out the spoiler effect (vote splitting)
- Plurality rule, for instance, does not satisfy WARP

| # | 60 | 50 |
|---|---|---|
| 1st | $x_1$ | $x_2$ |
| 2nd | $x_2$ | $x_1$ |

$\rightarrow$

| # | 15 | 45 | 50 |
|---|---|---|---|
| 1st | $x_3$ | $x_1$ | $x_2$ |
| 2nd | $x_1$ | $x_3$ | $x_1$ |
| 3rd | $x_2$ | $x_2$ | $x_3$ |

# Condorcet consistency

**Condorcet winner**

- An alternative which wins the pairwise-majority voting against every other is called a Condorcet winner
- Condorcet winner does not always exist (See table)
- A voting rule is said to be Condorcet-winner-consistent if it always chooses a Condorcet winner (if exists)

| # | 1 | 1 | 1 |
|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ |
| 2nd | $x_2$ | $x_3$ | $x_1$ |
| 3rd | $x_3$ | $x_1$ | $x_2$ |

# Condorcet consistency (cont'd)

**Condorcet loser**

- An alternative which loses the pairwise-majority voting against every other is called a Condorcet loser
- For example, $x_1$ in the table below is a Condorcet loser
- A voting rule is said to be Condorcet-loser-consistent if it never chooses a Condorcet loser

| # | 4 | 4 | 7 | 6 |
|---|---|---|---|---|
| 1st | $x_1$ | $x_1$ | $x_2$ | $x_3$ |
| 2nd | $x_2$ | $x_3$ | $x_3$ | $x_2$ |
| 3rd | $x_3$ | $x_2$ | $x_1$ | $x_1$ |

# Condorcet consistency (cont'd)

**Condorcet-winner consistency (CWC)**

- CWC is quite a strong requirement
- In fact only a few voting rules can satisfy this criterion
- For example, it should be easy to see that tournament rule satisfies CWC (does not satisfy unanimity, though)

**Condorcet-loser consistency (CLC)**

- CLC is a relatively weak requirement
- Some of the commonly used voting rules, however, do not satisfy this criterion
- For instance, plurality rule, which is perhaps the most widely used voting rule in the world, fails to satisfy CLC

# Condorcet consistency (cont'd)

**Plurality rule fails to satisfy CLC**

- Consider a preference profile listed in the table below (which we have already seen in a preceding slide)
- Observe that $x_1$ is a Condorcet loser, which should not be chosen by any good voting rule
- But plurality rule chooses $x_1$!

| # | 4 | 4 | 7 | 6 |
|---|---|---|---|---|
| 1st | $x_1$ | $x_1$ | $x_2$ | $x_3$ |
| 2nd | $x_2$ | $x_3$ | $x_3$ | $x_2$ |
| 3rd | $x_3$ | $x_2$ | $x_1$ | $x_1$ |

# Binary independence

**Binary independence (BI)**

- Also known as 'independence from irrelevant alternatives (IIA)'
- Pairwise (binary) comparison of two alternatives should be independent of how other alternatives are ranked:

$$\succsim_i = \succsim_i' \text{ on } \{x_1, x_2\} \text{ for all } i \implies f(\rho, \{x_1, x_2\}) = f(\rho', \{x_1, x_2\}) \tag{5}$$

- As WARP, intended to rule out the vote splitting

**Relevance**

- BI seems reasonable (innocuous at least) at first glance
- But this in fact is a very strong requirement
- If we stick to BI, almost no voting rule is possible (the result known as Arrow's impossibility theorem)

# Binary independence (cont'd)

**Spoiler effect (vote splitting)**

- Consider preference profiles $\succsim$ (left) and $\succsim'$ (right) listed in the tables below
- If $f : \mathcal{R}^n \times \mathcal{S} \to X$ is the plurality rule, then we have

$$f(\rho, \{x_1, x_2\}) = x_1 \quad \text{but} \quad f(\rho', \{x_1, x_2\}) = x_2,$$

which violates the binary independence criterion

| # | 60 | 50 |
|---|---|---|
| 1st | $x_1$ | $x_2$ |
| 2nd | $x_2$ | $x_1$ |
| 3rd | $x_3$ | $x_3$ |

$\rightarrow$

| # | 15 | 45 | 50 |
|---|---|---|---|
| 1st | $x_3$ | $x_1$ | $x_2$ |
| 2nd | $x_1$ | $x_3$ | $x_1$ |
| 3rd | $x_2$ | $x_2$ | $x_3$ |

# Participation criterion

**No-show paradox**

- Hopefully, a good voting rule should not be susceptible to strategic participation
- Run-off rule, however, does not satisfies this criterion
- Some voters may achieve a more preferable outcome if they choose not to participate (See table)

| # | 5 | 4 | 6 |
|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ |
| 2nd | $x_2$ | $x_3$ | $x_1$ |
| 3rd | $x_3$ | $x_1$ | $x_2$ |

$\rightarrow$

| # | 3 | 4 | 6 |
|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ |
| 2nd | $x_2$ | $x_3$ | $x_1$ |
| 3rd | $x_3$ | $x_1$ | $x_2$ |

# Preference revealing

**Misrepresenting preferences**

- It is known to be quite difficult to design a voting rule in such a way that nobody has an incentive to tell a lie
- Under the Borda rule, for instance, voters can manipulate the outcome by false revelation (See table)

| # | 1 | 1 | 1 |
|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ |
| 2nd | $x_2$ | $x_1$ | $x_2$ |
| 3rd | $x_3$ | $x_4$ | $x_1$ |
| 4th | $x_4$ | $x_3$ | $x_4$ |

$\rightarrow$

| # | 1 | 1 | 1 |
|---|---|---|---|
| 1st | $x_1$ | $x_2$ | $x_3$ |
| 2nd | $x_4$ | $x_1$ | $x_2$ |
| 3rd | $x_3$ | $x_4$ | $x_1$ |
| 4th | $x_2$ | $x_3$ | $x_4$ |

# Digression: Ostrogorski paradox

**Candidate selection or issue-by-issue voting**

- Suppose there are three issues at stake $A, B, C$, on which two candidates $x_1, x_2$ take different positions
- Which voting procedure would be a better choice: candidate selection or issue-by-issue voting?
- Perhaps neither of them is satisfactory for the voters

| # | 3 | 2 |
|---|---|---|
| 1st | $x_1$ | $x_2$ |
| 2nd | $x_2$ | $x_1$ |

or

| # | 1 | 1 | 1 | 2 |
|---|---|---|---|---|
| $A$ | $x_1$ | $x_1$ | $x_2$ | $x_2$ |
| $B$ | $x_1$ | $x_2$ | $x_1$ | $x_2$ |
| $C$ | $x_2$ | $x_1$ | $x_1$ | $x_2$ |