A theory of disasters and long-run growth

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Abstract

This paper develops a unified framework in which various types of catastrophic shocks can be simultaneously considered within a standard model of economic growth. We first establish the basic existence and equivalence results. We then apply the framework to an endogenous growth model to consider the influence of disasters on the long-term equilibrium and the transition phase. The result shows that while experiencing disasters may lower the average growth rate of the affected countries, there exist various channels through which the risk of disasters and long-term economic performance are positively correlated. This finding reconciles the apparently contradictory evidence in recent empirical studies.

Keywords: disasters; dynamic optimization; long-term growth; endogenous growth; aggregate uncertainty

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1 Introduction

Our economy is facing the risk of various disasters which, once they happen, undo many years of capital accumulation. Among the most obvious examples are natural disasters such as hurricanes, floods, and earthquakes. Hurricane Katrina of 2005, for instance, destroyed a significant amount of capital accumulated along the Gulf coast and caused the estimated cost of more than 100 billion dollars (Blake et al., 2011). Widespread epidemics of infectious diseases, although not so destructive in terms of physical capital, can also have a devastating impact on the accumulation of human capital. A classic example is the Black Death in the 14th century, which is estimated to have killed 30 to 60 percent of the total population in Europe (Alchon, 2003). Disasters can also be man-made, as is clearly demonstrated by nuclear explosions and military conflicts. The catastrophic accident in the Chernobyl nuclear power plant in 1986, for example, made the surrounding area uninhabitable for the next few hundreds years. During the period of World War II, by one estimate, 70 percent of the industrial infrastructure of Europe was demolished (Pilisuk and Rountree, 2008). More recently, increasing incidents of terrorist attacks have emerged as another type of man-made disasters, which could kill thousands of people and cost billions of dollars in infrastructure as in the September 11 attacks on the United States in 2001. Whatever their origin, disasters drastically change the state of our economy and therefore have potentially significant consequences to the long-run growth of the economy.

In spite of its importance, only fragmented efforts have been made to incorporate the risk of disasters into the standard marcoeconomic framework. In the macro-finance literature, the impact of short-run economic disasters has been studied by Barro (2006, 2009) and Gabaix (2012). The focus of these studies, however, is on the interplay between rare macroeconomic disasters and asset-pricing puzzles. Their analysis is based on variants of the endowment-economy model of Lucas (1978) and so far, long-term consequences of disasters have not been well incorporated (Barro and Ursua, 2012). In the economic growth literature, Ikefuji and Horii (2012) analyze the role of natural disasters in the two-sector endogenous growth model of Uzawa (1965) and Lucas (1988). But their primary interest is in pollution-induced disasters and pollution control, not in the disaster-growth relationship *per se*. Also, in their framework, the risks of disasters are modeled as idiosyncratic shocks to capital stocks and stochasticity is largely missing at the aggregate level. In a different strand of growth literature, economic consequences of wars and epidemics have been occasionally discussed, often in relation to imbalance effects. For instance, Mulligan and Sala-i-Martin (1993) show that an economy that loses a lot of physical infrastructure in a war will recover quickly, whereas an economy that loses a large fraction of human capital in an epidemic will suffer sluggish growth along the transition path to the long-run equilibrium. In their analysis, however, disasters are modeled as exogenous changes in the initial condition and therefore the risk of disasters is not taken into account in the decision maker's optimization problem.

The objective of the present paper is to provide a unified framework in which various types of disasters can be simultaneously considered within a standard model of economic growth. Our analysis starts with dynamic optimization theory and introduces the risk of disasters within a canonical discrete-time infinite-horizon optimization problem. We then establish a basic existence result for the stochastic optimization problem in a fairly general form. In the deterministic optimization problem, existence and uniqueness results have been discussed by Alvarez and Stokey (1998) and Le Van and Morhaim (2002), among others. We take advantage of these existing results by not directly addressing the stochastic optimization problem, but rather reformulating it into an 'equivalent' deterministic problem. Once we reformulate the original stochastic problem as an associated deterministic problem, we can apply the well-established tools of deterministic optimization theory in analyzing the solution of the stochastic problem. This strategy allows us to analyze catastrophic shocks of various kinds in a transparent fashion, better understand their long-run macroeconomic consequences, and formulate simple yet effective lines of arguments. By applying the general framework to an endogenous growth model à la Lucas (1988), we examine in detail the relationship between disasters and growth. The existence and uniqueness of a balanced growth path is readily established, along with the necessary and sufficient condition. We fully characterize the long-term equilibrium as well as the transition phase.

The highly transparent nature of our approach is particularly useful when it is matched up with the existing and growing body of empirical studies. Empirical observations about the disaster-growth relationship date back to a famous writing of Mill (1848), but substantive research on this topic has only been made recently. Somewhat surprisingly, the recent empirical findings are quite inconclusive, especially in the context of natural disasters. A seminarl study by Skidmore and Toya (2002) finds that the frequency of climatic disasters[1] is positively correlated with long-term economic growth. A higher frequency of disasters also facilitates human capital accumulation and productivity growth. Their results, which are based on a cross-country analysis over the period between 1960 and 1990, provide a stark contrast to other empirical studies such as Raddatz (2007) and Noy (2009). These studies demonstrated that the short-run economic impacts that directly follow disasters are generally negative. More recently, by focusing on countries' exposure to tropical cyclones during the 1950-2008 period, Hsiang and Jina (2014, 2015) present evidence that national incomes decline, relative to their pre-disaster trend, and do not recover within twenty years.[2]

The apparently contradictory empirical evidence about the disaster-growth relationship can be at least partially reconciled by our theoretical analysis. Our result shows that while actual *experience* of disasters may lower the average growth rate in the affected countries, there are various channels through which the *risk* of disasters and long-term economic performance are positively correlated.[3] The analyses of Raddatz (2007), Noy (2009) and Hsiang and Jina (2014, 2015) capture the impact of disaster strikes by comparing the economic performance of affected countries against a counterfactual scenario of no disaster experience. The positive correlation reported by Skidmore and Toya (2002), on the other hand, is based on a cross-country study and, therefore, reflects the potentially growth-enhancing effect of disaster risks.

Our analysis also clarifies how exactly disasters affect economic growth. One suggested explanation for the existing empirical findings is that disasters affect the relative return on capital investment (Skidmore and Toya, 2002). In turn, this explanation is based on the fact that some types of disasters are primarily destructive to physical capital. For instance, the damage from storms is intensive in terms of physical capital, whereas extreme temperatures or droughts have a greater effect on human capital.[4] If the impact of disasters is biased against

[1]In their study, this category of disasters includes floods, cyclones, hurricanes, ice storms, snow storms, tornadoes, typhoons, and storms.

[2]Apart from natural disasters, Hirshleifer (1987) documents in detail the consequence of and the recovery process from wars and epidemics. Blomberg et al. (2004) presents a unique empirical analysis of the macroeconomic consequences of terrorism.

[3]Similar to our paper, Bakkensen and Barrage (2016) analyze a growth model and note the importance of distinction between disaster strikes and risk. In many ways, their analysis is complementary to ours. They use a particular class of stochastic growth model and characterize the behavior of decentralized economy in detail. Our analysis, on the other hand, centers around the general optimal growth model where additional dimensions of the economy (such as the elasticity of substitution in the production function) can be taken into account.

[4]See Guha-Sapir et al. (2013) for an extensive review of different types of disasters.

physical capital, the relative return on capital investment may be tilted in favor of human capital, which is likely to boost economic growth. In addition, if some inefficiency remains in the economy, the destruction of physical capital itself can improve long-term economic performance. By destroying old factories and roads, disasters allow new and more efficient infrastructure to be built, providing an opportunity for the economy to transform itself into a more productive one in the long run. Our analysis shows that as long as the damage is restricted to the stock of physical capital, the long-term growth rate may not be affected at all. We regard this result as providing theoretical support for the current empirical findings. Given that the degradation of efficiently accumulated physical capital is at least not harmful, the destruction of inefficiently invested physical capital could improve the growth rate in the long run. Moreover, even if there is no inefficiency, physically destructive disasters can improve the economic growth rate when the economy remains in transition to the long-term equilibrium. This result captures another suggested channel in which a change in relative return plays an important role. However, we find that this rate-of-return effect crucially depends on the substitutability between physical capital and effective labor. Hence, in future empirical studies, the technological characteristics of the economy should be taken into account, along with the type of disaster.

In addition, our theoretical framework allows us to identify yet another possible driver behind the supposed disaster-growth relationship, namely, the aggregate uncertainty matters. If the timing of a large-scale disaster is unpredictable and uninsurable, economic growth may be sped up, even if the damage is not biased against physical capital. Faced with potential disasters in the future, the optimal policy requires that the available resources should be reallocated from the accumulation of physical capital to the development of human capital. This results from the consumption-smoothing motive in that more frequent disasters lower the expected level of future consumption. Unless consumption is highly substitutable over time, such a dismal expectation is counteracted by investing more in human capital and saving more for future consumption. This precautionary savings effect then spurs economic growth in the long run. What is indicated by this finding is that the high economic growth rate resulting from disasters may only be achieved at the cost of suppressed consumption by the present generation.

Our analysis also presents a new perspective on cross-country differences in long-run economic performance. It is widely known that economic growth rates

Table 1: Major natural disasters in different countries (1960–2012)

	Total	Type (count)
United States	770	Storm (484), Flood (148), Wildfire (60)
China	664	Storm (216), Flood (206), Earthquake (121)
India	560	Flood (234), Storm (137), Epidemic (63)
Philippines	521	Storm (283), Flood (131), Landslide (30)
Indonesia	405	Flood (151), Earthquake (99), Volcanic (46)
Bangladesh	299	Storm (148), Flood (85), Epidemic (29)
Japan	228	Storm (116), Earthquake (40), Flood (35)
Mexico	216	Storm (82), Flood (57), Earthquake (27)
Australia	209	Storm (98), Flood (59), Wildfire (28)
Russia	198	Flood (69), Earthquake (28), Storm (24)
Brazil	197	Flood (114), Landslide (21), Drought (17)
Viet Nam	179	Storm (88), Flood (68), Epidemic (10)
Iran	178	Earthquake (90), Flood (66), Storm (12)

Source: EM-DAT, the OFDA/CRED International Disaster Database, Université Catholique de Louvain, Brussels, Belgium.

differ significantly across different countries.[5] While the observed difference is largely attributed to the different levels of physical and human capital between countries, the fundamental cause that underlies this observation is not well understood. Given the fact that countries are exposed to different types of disasters of dissimilar frequency (See, for example, Table 1 for the major natural disasters experienced by different countries.), establishing the formal link between disasters and economic growth opens up the possibility of partly explaining these cross-country differences at a fundamental level.

The rest of the paper is organized as follows. Section 2 develops the general framework and presents the existence and equivalence results. In Section 3, we describe how the framework can be applied to the endogenous growth model. We prove the existence and uniqueness of the balanced growth path and discuss its properties. In particular, we highlight the impact of disasters on the long-run growth rate. We also examine the economy off the balanced growth path and clarify how the transition phase is influenced by the presence of disasters. In Section 4, we discuss robustness of our result and possible extensions for future research. Section 5 concludes the paper.

[5]See, for example, Acemoglu (2008) for a concise review.

2 The model

We consider an infinite-horizon discrete-time model. Periods are indexed by $t \in \mathbb{Z}_{++} := \{1, 2, ...\}$. By period t, we mean the time interval from $t-1$ up to t. We describe the economy in an aggregate fashion. The set of all feasible stock paths is characterized by transition correspondence $\boldsymbol{y} \in \Gamma(\boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{R}^n_+$ and $\boldsymbol{y} \in \mathbb{R}^n_+$ are n-dimensional vectors of capital stocks at the beginning and the end of each period, respectively. We denote the one-period return function by $R(\boldsymbol{x}, \boldsymbol{y})$, which is weighted over time by the discount factor $\beta \in (0, 1)$. The dynamic optimization problem of this form has been extensively studied in the literature. See Le Van (2006) for a comprehensive treatment. We extend this general framework by considering two distinct aspects of disasters separately: unpredictability of disaster strikes and potential bias against specific types of capital.

2.1 Aggregate risk

Some types of disasters catch us by surprise. They are generally unpredictable and, once they occur, have substantial impacts on the economy through the destruction of capital stock.[6] To capture this feature, we introduce an aggregate risk of capital destruction, the occurrence of which follows a Bernoulli process with probability $\lambda \in (0, 1)$. In order to disentangle the role of unpredictability from the role of capital-specific bias (which we will introduce later), we assume that unpredictable disasters destroy all types of capital by the same proportion. To be more precise, only a fraction $\alpha \in (0, 1)$ of the capital stock survives each occurrence of disaster so that the capital stock vector becomes $\alpha\boldsymbol{x}$ instead of $\boldsymbol{x}$.

A decision is made in each period after the current uncertainty is resolved. If $\boldsymbol{y}_t$ is the capital stock at the end of period t, for instance, the economy at the beginning of period $t+1$ has $\boldsymbol{x}_{t+1} = D_t\boldsymbol{y}_t$, where $D_t \in \{1, \alpha\}$ is a random variable following the Bernoulli process. Decision making is contingent upon a realized path of disasters. We denote the history of disasters up until period t by $D^t := (D_1, D_2, ..., D_t) \in \{1, \alpha\}^t$.

[6] The type of disasters that we are considering here is loosely connected to what Hirshleifer (1991) calls the society-wide (large-scale) catastrophes. On top of the society-wide disasters, he also considers more frequent, community-wide (middle-scale) calamities, that in our framework is called the idiosyncratic risk. But unlike Hirshleifer (1991), we do not think that the real-world disasters can be actually categorized this way. The dichotomy between aggregate and idiosyncratic shocks is only for the sake of transparent analysis, intended to capture different aspects of each disaster.

Let X be the x-projection of the effective domain of R given Γ. To be more precise, X is a subset of $\mathbb{R}^n_+$ such that $\boldsymbol{x} \in X$ implies $R(\boldsymbol{x}, \boldsymbol{y}) > -\infty$ for some $\boldsymbol{y} \in \Gamma(\boldsymbol{x})$. The dynamic optimization problem is then formulated as

$$\begin{aligned} V(\boldsymbol{x}) := \sup_{\{\boldsymbol{y}_t\}_{t=1}^{\infty}} \; & \mathbb{E}\left[R(\boldsymbol{x}, \boldsymbol{y}_1) + \sum_{t=1}^{\infty} \beta^t R(D_t \boldsymbol{y}_t(D^{t-1}), \boldsymbol{y}_{t+1}(D^t))\right] \\ \text{s.t.} \quad & \boldsymbol{y}_{t+1}(D^t) \in \Gamma(D_t \boldsymbol{y}_t(D^{t-1})) \text{for all } D^t \text{ and } t \in \mathbb{Z}_{++}, \\ & \boldsymbol{y}_1 \in \Gamma(\boldsymbol{x}) \end{aligned} \tag{1}$$

for each $\boldsymbol{x} \in X$, where $\mathbb{E}$ is the expectation operator and we put $\boldsymbol{y}_1(D^0) := \boldsymbol{y}_1$. This planning problem is only meaningful if the optimal solution exists. Hence, we first need to show that problem (1) has a solution. Nonetheless, directly working with the stochastic optimization problem requires complicated assumptions.

To establish the existence result in a transparent fashion, we reformulate the stochastic problem into a much simpler deterministic form. We can achieve this by introducing a couple of innocuous assumptions. To ease the notation, we denote the graph of Γ by

$$\operatorname{graph}(\Gamma) := \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \mid \boldsymbol{y} \in \Gamma(\boldsymbol{x})\}. \tag{2}$$

Assumption 1. $\Gamma : \mathbb{R}^n_+ \rightrightarrows \mathbb{R}^n_+$ is nonempty-valued with $\Gamma(\mathbf{0}) = \{\mathbf{0}\}$. Also, $(\boldsymbol{x}, \boldsymbol{y}) \in \operatorname{graph}(\Gamma)$ implies $(\gamma\boldsymbol{x}, \gamma\boldsymbol{y}) \in \operatorname{graph}(\Gamma)$ for any $\gamma > 0$.

Assumption 2. $R : \operatorname{graph}(\Gamma) \to \mathbb{R} \cup \{-\infty\}$ is homogeneous of degree $\theta < 1$.

Assumption 1 requires that the graph of Γ be a cone, which is consistent with many economic models. The homogeneity requirement in Assumption 2 is also satisfied by the commonly used class of utility functions. Under these assumptions, the stochastic optimization problem (1) can be rewritten as an associated deterministic problem.

Lemma 2.1 (Value function equivalence)**.** *Under Assumptions 1 and 2, V defined in* (1) *satisfies*

$$V(\boldsymbol{x}) = \sup\left\{ \sum_{t=1}^{\infty} \tilde{\beta}^{t-1} R(\tilde{\boldsymbol{y}}_{t-1}, \tilde{\boldsymbol{y}}_t) \;\middle|\; \tilde{\boldsymbol{y}}_t \in \Gamma(\tilde{\boldsymbol{y}}_{t-1}),\ \tilde{\boldsymbol{y}}_0 = \boldsymbol{x} \right\} \tag{3}$$

for each $\boldsymbol{x} \in X$, where

$$\tilde{\beta} := (\lambda\alpha^{\theta} + 1 - \lambda)\beta. \tag{4}$$

The proof is tedious and is, therefore, relegated to Appendix A.1. One can intuitively understand this result by noticing its connection to the classic result of Yaari (1965). Consider, for instance, an extreme case of disaster (or 'the end of life') such that once it happens, the value of the return function will be 0 thereafter. In such a case, the decision maker effectively discounts her per-period return function by $(1-\lambda)\beta \approx e^{-\lambda}\beta$. In other words, the effective discount rate is raised by the hazard rate, a well-known result since Yaari (1965). In our model, Yaari's analysis corresponds to the case where $\alpha = 0$, which means that the stock of capital completely collapses when a disaster hits the economy. Lemma 2.1 generalizes this well-known result to the cases where the consequence of disasters is not necessarily catastrophic and an incident of disaster is not a one-off event.[7] Our formula (4) suggests that, in general, the effective discount rate is not only a function of the hazard rate, but also of the magnitude of the disaster as well as the degree of homogeneity of the return function. As we will see shortly, this fact is crucial when examining the relationship between growth and disasters.

The equivalence result obtained in Lemma 2.1 makes our task significantly easier. Since the deterministic problem (3) is well understood in the literature, we can now apply the powerful tools established in the deterministic optimization theory in analyzing the solution of the original stochastic problem (1). Our analysis hereafter is thus primarily focused on the deterministic formulation (3). Accordingly, we say that a path $\{\tilde{\boldsymbol{y}}_t\}_{t=1}^{\infty}$ is feasible from $\boldsymbol{x} \in X$ if $\tilde{\boldsymbol{y}}_t \in \Gamma(\tilde{\boldsymbol{y}}_{t-1})$ for all $t \in \mathbb{Z}_{++}$ with $\tilde{\boldsymbol{y}}_0 = \boldsymbol{x}$. Here we put tildes on the variables in order to emphasize the fact that they represent a path in the deterministic problem, not in the original stochastic problem. We will later discuss how the optimal paths of the two different formulations are related.

Now that the problem is given as a familiar deterministic optimization problem, the existence result is in order. Before proving the existence of the optimal path, we note that the effective discount factor $\tilde{\beta}$ defined in (4) can be greater than unity when $\theta < 0$. This does not cause a problem in the present context because, as in the case of a homogeneous utility function, the return function is assumed to be bounded from above when θ is negative. With this remark in

[7] In the field of pollution control and resource management, Yaari's framework has been extended in various directions. Clarke and Reed (1994), for example, considered an irreversible catastrophic event whose hazard rate is influenced by the stock of pollution. Their model was later extended by Tsur and Zemel (1998) to allow for multiple occurrences of an environmental event. In this literature, the consequence of 'disaster' is usually modeled as a flow penalty or a sudden change of the model's primitives, not as a destruction of existing capital.

mind, we can appreciate that the following assumptions are reminiscent of those in Le Van (2006):

Assumption 3. Γ is compact-valued and continuous.

Assumption 4. graph(Γ) is convex, R is smooth on the interior of graph(Γ), $\theta \neq 0$, and

(a) if $\theta > 0$, $R(\boldsymbol{y}, \boldsymbol{y}') > 0$ on the interior of graph(Γ);

(b) if $\theta < 0$, $R(\boldsymbol{y}, \boldsymbol{y}') < 0$ on graph(Γ);

(c) R is continuous in the extended sense, namely, if $R(\boldsymbol{y}, \boldsymbol{y}') = -\infty$, then $\lim_{n\to\infty} R(\boldsymbol{y}_n, \boldsymbol{y}'_n) = -\infty$ for any sequence $\{(\boldsymbol{y}_n, \boldsymbol{y}'_n)\}_{n=1}^{\infty} \subset$ graph(Γ) such that $(\boldsymbol{y}_n, \boldsymbol{y}'_n) \to (\boldsymbol{y}, \boldsymbol{y}')$ as $n \to \infty$.

Assumption 5. For each $\boldsymbol{y}_0 \in X$,

(a) if $\theta > 0$, $\sum_{t=1}^{\infty} \tilde{\beta}^{t-1} R(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t) < \infty$ for any feasible path from $\boldsymbol{y}_0$;

(b) if $\theta < 0$, $\sum_{t=1}^{\infty} \tilde{\beta}^{t-1} R(\boldsymbol{y}_{t-1}, \boldsymbol{y}_t) > -\infty$ for some feasible path from $\boldsymbol{y}_0$.

Assumption 3 is standard, which ensures that the set of all feasible paths is compact in the product topology. Assumption 4 combined with Assumption 2 imply that R is concave on graph(Γ) and strictly concave on the interior of graph(Γ)).[8] With Assumption 5, the objective function in (3) is upper semicontinuous in the product topology on the set of all feasible paths.[9] We now present the following basic result.

Proposition 2.1 (Policy function equivalence)**.** *Under Assumptions 1–5, i) there is an optimal policy function $\psi : X \to X$ of the deterministic problem (3), ii) V is homogeneous of degree θ, iii) ψ is homogeneous of degree* 1*, and iv) ψ is an optimal policy function of the original stochastic problem (1). Conversely, v) any optimal policy function of the stochastic problem (1) solves the deterministic problem (3).*

[8]Fix $\boldsymbol{z} \in \text{int(graph}(\Gamma))$ and twice differentiate both sides of $R(\gamma \boldsymbol{z}) = \gamma^{\theta} R(\boldsymbol{z})$ with respect to γ. Evaluated at $\gamma = 1$, this yields $\langle \boldsymbol{z}, \left(d^2 R(\boldsymbol{z})/d\boldsymbol{z}^2\right) \boldsymbol{z} \rangle = \theta(\theta - 1) R(\boldsymbol{z}) < 0$, where $\langle \cdot, \cdot \rangle$ is the inner product. This means that the Hessian of R is negative definite, which implies the strict concavity of R on the interior. The concavity on the entire domain then follows from the continuity of R.

[9]See Lemma 2.2.1 of Le Van (2006).

Proof. i) The existence of ψ follows from Proposition 2.2.1 of Le Van (2006). ii) Fix $\boldsymbol{x} \in X$. Notice that for any $\gamma > 0$, whenever $\{\tilde{\boldsymbol{y}}_t\}_{t=1}^{\infty}$ is feasible from $\boldsymbol{x}$, so is $\{\gamma\tilde{\boldsymbol{y}}_t\}_{t=1}^{\infty}$ from $\gamma\boldsymbol{x}$. Hence, we must have $V(\gamma\boldsymbol{x}) \geq \gamma^{\theta}V(\boldsymbol{x})$. To see the reverse inequality, let $\boldsymbol{X}(\boldsymbol{x})$ be the set of all feasible paths starting from $\boldsymbol{x}$. Suppose, by way of contradiction, that for some $\gamma > 0$, there exists $\{\tilde{\boldsymbol{y}}_t\}_{t=1}^{\infty} \in \boldsymbol{X}(\gamma\boldsymbol{x})$ such that $\sum_{t=1}^{\infty}\tilde{\beta}^{t-1}R(\tilde{\boldsymbol{y}}_{t-1}, \tilde{\boldsymbol{y}}_t) > \gamma^{\theta}V(\boldsymbol{x})$. Then by homogeneity of R, we have $\sum_{t=1}^{\infty}\tilde{\beta}^{t-1}R(\gamma^{-1}\tilde{\boldsymbol{y}}_{t-1}, \gamma^{-1}\tilde{\boldsymbol{y}}_t) > V(\boldsymbol{x})$. Since $\{\gamma^{-1}\tilde{\boldsymbol{y}}_t\}_{t=1}^{\infty} \in \boldsymbol{X}(\boldsymbol{x})$ by homogeneity of Γ, however, this contradicts the fact that V is the value function of (3). It follows that $V(\gamma\boldsymbol{x}) = \gamma^{\theta}V(\boldsymbol{x})$ for any $\gamma > 0$. iii) Fix $\boldsymbol{x} \in X$ and let $\{\tilde{\boldsymbol{y}}_t\}_{t=1}^{\infty} \in \boldsymbol{X}(\boldsymbol{x})$ be the optimal path generated by $\tilde{\boldsymbol{y}}_{t+1} = \psi(\tilde{\boldsymbol{y}}_t)$ for each t. Then $\{\gamma\tilde{\boldsymbol{y}}_t\}_{t=1}^{\infty}$ is feasible from $\gamma\boldsymbol{x}$ for any $\gamma > 0$ and

$$\sum_{t=1}^{\infty}\tilde{\beta}^{t-1}R(\gamma\tilde{\boldsymbol{y}}_{t-1}, \gamma\tilde{\boldsymbol{y}}_t) = \gamma^{\theta}\sum_{t=1}^{\infty}\tilde{\beta}^{t-1}R(\tilde{\boldsymbol{y}}_{t-1}, \tilde{\boldsymbol{y}}_t) = \gamma^{\theta}V(\boldsymbol{x}) = V(\gamma\boldsymbol{x}), \tag{5}$$

where the last equality follows from result ii). This immediately implies that $\psi(\gamma\boldsymbol{x}) = \gamma\psi(\boldsymbol{x})$ for any $\gamma > 0$ and $\boldsymbol{x} \in X$. iv) Given that V in (3) is well defined, it satisfies the Bellman equation

$$V(\boldsymbol{x}) = \max_{\tilde{\boldsymbol{y}}\in\Gamma(\boldsymbol{x})}\left\{R(\boldsymbol{x}, \tilde{\boldsymbol{y}}) + \tilde{\beta}V(\tilde{\boldsymbol{y}})\right\} \tag{6}$$

for each $\boldsymbol{x} \in X$. By Lemma 2.1 above, V is also the value function of the original stochastic problem (1). This means that V also satisfies the following Bellman equation:

$$V(\boldsymbol{x}) = \max_{\boldsymbol{y}\in\Gamma(\boldsymbol{x})}\left\{R(\boldsymbol{x}, \boldsymbol{y}) + \beta\mathbb{E}[V(D\boldsymbol{y})]\right\}. \tag{7}$$

Since V is homogeneous of degree θ,

$$\beta\mathbb{E}[V(D\boldsymbol{y})] = \beta\left(\lambda V(\alpha\boldsymbol{y}) + (1-\lambda)V(\boldsymbol{y})\right) = \tilde{\beta}V(\boldsymbol{y}), \tag{8}$$

which implies that (7) and (6) are equivalent for each $\boldsymbol{x} \in X$. Therefore, ψ is an optimal policy function of (1) as well. v) Immediate from (6), (7), and (8). $\square$

This result is quite useful. It states that equivalence holds between the deterministic and stochastic formulations, not only in terms of the value function, but also for the policy function. Focusing on the optimal policy of the deterministic problem, therefore, proves to be sufficient for the characterization of the optimal path of the original stochastic problem. We should mention, however, that a *realization* of the optimal path in the stochastic model, which depends on the corresponding realization of disaster history, does not necessarily replicate the deterministic optimal path. We will return to this issue later.

2.2 Idiosyncratic risk

Different types of disasters may destroy different kinds of production factors. For example, some types of disasters, such as droughts, may be particularly destructive in terms of human capital. Others, such as storms, may be more devastating in terms of physical capital. Since countries are exposed to different types of disasters of dissimilar frequency, it should be worthwhile to investigate how potentially biased damages of disasters affect the long-term growth rate of the economy. To this end, in addition to the economy-wide aggregate risk of disasters introduced above, we consider a smaller scale, idiosyncratic risk of disasters for each type of capital.

Let $z_i \in [0,1]$ be the fraction of capital $i \in \{1,2,\dots,n\}$ which survives strikes of small-scale disasters within each period. Since the occurrence and the magnitude of disasters are usually uncertain, the fraction z_i is a random variable at the firm or household level. Assuming that the risk of small-scale disasters is idiosyncratic across time and location within each economy, however, the law of large numbers suggests that their influence at the aggregate level can be captured by the expected values $\zeta_i := \mathbb{E}[z_i] \in (0,1)$. If $\tilde{\boldsymbol{y}}_t = (\tilde{y}_{1,t}, \tilde{y}_{2,t}, \dots, \tilde{y}_{n,t})^\top$ is the capital stock at the end of period t, for instance, the economy at the beginning of period $t+1$ has $\tilde{\boldsymbol{x}}_{t+1} = \boldsymbol{\zeta}\tilde{\boldsymbol{y}}_t = (\zeta_1\tilde{y}_{1,t}, \zeta_2\tilde{y}_{2,t}, \dots, \zeta_n\tilde{y}_{n,t})^\top$, where $\boldsymbol{\zeta}$ is an $n \times n$ diagonal matrix whose (i,i) element is ζ_i for each $i = 1,2,\dots,n$ and 0 everywhere else. This process is, at the aggregate level, seen as deterministic.[10] In other words, the idiosyncratic risk captures the rate-of-return effect of disasters only, while the unpredictability of disasters is taken into account by the aggregate risk that we discussed above.

Our problem is now reduced to the dynamic optimization problem

$$\begin{aligned} V(\boldsymbol{x}) &= \max\left\{ \sum_{t=1}^{\infty} \tilde{\beta}^{t-1} R(\boldsymbol{\zeta}\tilde{\boldsymbol{y}}_{t-1}, \tilde{\boldsymbol{y}}_t) \;\middle|\; \tilde{\boldsymbol{y}}_t \in \Gamma(\boldsymbol{\zeta}\tilde{\boldsymbol{y}}_{t-1}), \tilde{\boldsymbol{y}}_0 = \boldsymbol{\zeta}^{-1}\boldsymbol{x} \right\} \\ &= \max\left\{ \sum_{t=1}^{\infty} \tilde{\beta}^{t-1} \tilde{R}(\tilde{\boldsymbol{y}}_{t-1}, \tilde{\boldsymbol{y}}_t) \;\middle|\; \tilde{\boldsymbol{y}}_t \in \tilde{\Gamma}(\tilde{\boldsymbol{y}}_{t-1}), \tilde{\boldsymbol{y}}_0 = \boldsymbol{\zeta}^{-1}\boldsymbol{x} \right\}, \end{aligned} \tag{9}$$

[10]The same observation is made by Ikefuji and Horii (2012) in the Uzawa-Lucas model.

where

$$\tilde{R}(\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}') := R(\boldsymbol{\zeta}\tilde{\boldsymbol{y}}, \tilde{\boldsymbol{y}}'), \quad \tilde{\Gamma}(\tilde{\boldsymbol{y}}) := \Gamma(\boldsymbol{\zeta}\tilde{\boldsymbol{y}}). \tag{10}$$

Notice that Lemma 2.1 and Proposition 2.1 remain valid as long as $\tilde{R}$ and $\tilde{\Gamma}$ satisfy the assumptions above. Introducing the risk of disasters in this general form has at least two advantages. First, we can see how the presence of disaster risks affects the optimal path without specifying the structure of the economy. The aggregate risk of disasters influences the effective discount rate through $\tilde{\beta}$. The idiosyncratic (and capital-specific) risk, in effect, alters the technology of the economy through $\tilde{R}$ and $\tilde{\Gamma}$. This mechanism is fairly general and should remain valid for a wide class of economic models. How each channel of the disaster-economy interplay plays out, however, depends on the structure of the underlying model. Second, since many of the well-known models of economic growth can be written as (9), it is fairly straightforward to investigate how exactly the influence of disaster is translated into the equilibrium growth rate in various growth models. In fact, we illustrate this point in the next section by applying our framework to an endogenous growth model à la Lucas (1988).[11]

Before moving on, it will be useful to reemphasize the fact that in our framework, two different *aspects* of disasters are modeled separately. We are not claiming that each of the real-world disasters can be actually categorized as either an aggregate risk or an idiosyncratic risk. What we call the aggregate risk will be best interpreted as an hypothetical construct that is intended to capture the unpredictability of disasters only. The idiosyncratic risk, on the other hand, should be viewed as representing the capital-specific bias of disasters only. It is this decomposition that allows us to analyze the role of disasters in a highly transparent manner. While this approach seems fairly reasonable, it is not without loss of generality. In particular, one could argue that the roles of disasters may not be decomposed this way and the unpredictability and the potential bias should be simultaneously taken into account as a single shock to the economy (i.e., the case where the value of α is differentiated across different types of capital). We believe that the economic consequence of such a shock can be approximately represented by a combination of the aggregate and idiosyncratic shocks. Nevertheless, one should be aware of what is potentially missing in our framework.

[11] In Appendix B.3, we briefly lay out the continuous-time analogue of the model and show that our findings all survive.

3 Endogenous growth example

Let $\boldsymbol{y}_t = (K_t, H_t)^\top$ be the vector of physical and human capital at the end of period t. Denote by $F(k, l)$ the production function of final goods, where k is the physical capital stock at the beginning of each period and l is effective labor. The total amount of raw labor input is assumed to be constant and normalized to one. Denoting the stock of human capital at the beginning of each period by h, effective labor is given by $l = (1-n)h$, where $1 - n \in [0, 1]$ is the fraction of raw labor used for the production of final goods. Letting C_t be consumption in period t, we write the accumulation dynamics of physical capital as

$$K_t = Y_t - C_t + (1 - \delta_K)k_t \tag{11}$$

with

$$Y_t = F(k_t, (1 - n_t)h_t), \tag{12}$$

where

$$k_t := \zeta_K D_t K_{t-1} \quad \text{and} \quad h_t := \zeta_H D_t H_{t-1} \tag{13}$$

are the stocks of physical and human capital at the beginning of period t.[12] Here, $\delta_K \in [0, 1]$ represents the depreciation rate of physical capital and (ζ_K, ζ_H) represents the expected fractions of capital surviving small-scale idiosyncratic disasters. The production function in the human capital sector is specified as $G(l) := \eta l$, where $l = nh$ is the effective labor input and $\eta > 0$ is the productivity coefficient. The accumulation process of end-of-period human capital is then governed by

$$\begin{aligned} H_t &= G(n_t h_t) + (1 - \delta_H)h_t \\ &= (\eta n_t + 1 - \delta_H)\, h_t, \end{aligned} \tag{14}$$

where $\delta_H \in [0, 1]$ is the depreciation rate of human capital. We specify the one-period utility as

$$u(C) := C^\theta / \theta \tag{15}$$

for some $\theta < 1$. Note that $1 - \theta > 0$ is the elasticity of marginal utility. In dynamic settings, a more sensible interpretation of θ is that $1/(1 - \theta)$ represents

[12] Recall that we use K_t and H_t for the end-of-period values.

the elasticity of intertemporal substitution.[13] For $\theta = 0$, the utility function is specified as $u(C) := \ln(C)$, which is obtained as the limit case of $(C^\theta - 1)/\theta$, an affine transformation of (15), for $\theta \to 0$. Although Assumption 2 is not satisfied by the logarithmic function, we can apply the same equivalence result by simply setting $\theta = 0$ and hence $\tilde{\beta} = \beta$. See Appendix B.1 for details.

We now consider the following stochastic optimization problem:

$$V(K, H) := \max_{\{C_t, n_t\}_{t=1}^\infty} \mathbb{E}\left[\sum_{t=1}^\infty \beta^{t-1} u(C_t)\right] \tag{16}$$

subject to (11), (12), and (14) and $(K_0, H_0) = (\zeta_K^{-1} K, \zeta_H^{-1} H)$. Defining the return function and the transition correspondence by

$$\tilde{R}(K, H, K', H') := \frac{\left(F\left(\zeta_K K, [1 + \eta^{-1}(1 - \delta_H)]\zeta_H H - \eta^{-1} H'\right) + (1 - \delta_K)\zeta_K K - K'\right)^\theta}{\theta}, \tag{17}$$

and

$$\begin{aligned}\tilde{\Gamma}(K, H) :=\Bigg\{ (K', H') \in \mathbb{R}_+^2 \,\Bigg|\, & F\left(\zeta_K K, [1 + \eta^{-1}(1 - \delta_H)]\zeta_H H - \eta^{-1} H'\right) + (1 - \delta_K)\zeta_K K \geq K' \geq (1 - \delta_K)\zeta_K K, \\ & (\eta + 1 - \delta_H)\zeta_H H \geq H' \geq (1 - \delta_H)\zeta_H H \Bigg\},\end{aligned} \tag{18}$$

respectively, this problem may be written as (1) where $R = \tilde{R}$ and $\Gamma = \tilde{\Gamma}$. Notice that we have not specified the production function F yet. To make the model fully consistent with the above assumptions, we require F to satisfy the following assumption:

Assumption 6. $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ is smooth, concave, and homogeneous of degree one with $F(0, 0) = 0$, $F_k > 0$, $F_l > 0$, $F_{kk} < 0$, and $F_{ll} < 0$.

It should be easy to verify that under Assumption 6, the growth model at hand satisfies Assumptions 1–4. Provided that Assumption 5 also holds, we can

[13] Parameter θ also reflects the degree of risk aversion because the Arrow-Pratt coefficient of risk aversion is given by $\gamma := 1 - \theta$. In Appendix B.4, we present a model with a more general class of utility function where risk aversion and elasticity of intertemporal substitution are disentangled. The analysis there suggests that in the present context, θ is best interpreted as capturing the consumption-smoothing motive, not the risk preference.

apply Proposition 2.1 and rewrite (16) as the deterministic growth model

$$\max_{\{(\tilde{K}_t,\tilde{H}_t)\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \tilde{\beta}^{t-1}\tilde{R}(\tilde{K}_{t-1},\tilde{H}_{t-1},\tilde{K}_t,\tilde{H}_t) \tag{19}$$
$$\text{s.t.} \quad (\tilde{K}_t,\tilde{H}_t) \in \tilde{\Gamma}(\tilde{K}_{t-1},\tilde{H}_{t-1}) \text{ and } (\tilde{K}_0,\tilde{H}_0) = (\zeta_K^{-1}K, \zeta_H^{-1}H).$$

For the sake of completeness, we provide a sufficient condition for Assumption 5.

Assumption 7. The production function satisfies

$$\lim_{k\to\infty} F_k(k,l) < \frac{\zeta_H}{\zeta_K}(\eta+1-\delta_H) - (1-\delta_K) < \lim_{k\to 0} F_k(k,l) \tag{20}$$

for all $k > 0$ and $l > 0$.

Proposition 3.1. *Under Assumptions 6 and 7, there exists an optimal policy function of the problem* (19) *if*

$$(\zeta_H\,[\eta+1-\delta_H])^{\theta}(\lambda\alpha^{\theta}+1-\lambda)\beta < 1. \tag{21}$$

This policy function also solves the problem (16)*.*

The proof is found in Appendix A.2. Assumption 7 is a variant of Inada condition, which requires that the marginal product of physical capital be sufficiently large when there is little stock of capital and become sufficiently small as the capital accumulates. The result of Proposition 3.1 is a reminiscent of Theorem 1 of Brock and Gale (1969), which show that the discount factor needs to be smaller than a critical level in order for the optimal solution to exist. Otherwise, there would be a feasible consumption plan with which the value of objective function is infinite. As we will show below, the condition (21) is not only sufficient, but also necessary if the solution is to be a balanced growth path.

3.1 Linking stochastic and deterministic solutions

As mentioned earlier, our discussion revolves around the deterministic formulation (19) rather than the original stochastic formulation (16). Before presenting the analysis, it will be useful if we clarify how the results of this section can be interpreted in the stochastic formulation.

As far as the growth rate is concerned, the optimal paths of the two formulations are identical, except for the periods when a disaster hits the economy. To see this, let $\{\tilde{C}_t, \tilde{K}_t, \tilde{H}_t\}_{t=1}^{\infty}$ be the optimal path of consumption and capital induced by the deterministic model and put $\tilde{g}_t := \tilde{C}_{t+1}/\tilde{C}_t$, the *gross* rate of consumption growth, for each $t \in \mathbb{Z}_{++}$. Since the two formulations of the model share the same policy function, their optimal paths exactly coincide until an economy-wide unpredictable disaster occurs. Suppose the economy then experiences a disaster at the beginning of period t. While this affects the *level* of each capital available in that period, the *ratio* is unaffected because $(\alpha\tilde{K}_t)/(\alpha\tilde{H}_t) = \tilde{K}_t/\tilde{H}_t$. Given that the graph of the transition correspondence is a cone, this implies that the optimal path of ratio of capitals remains the same whatever the realization $\{D_t\}_{t=1}^{\infty}$ of the disaster history will be. Consequently, the associated realization $\{C_t\}_{t=1}^{\infty}$ of the optimal consumption path in the stochastic model is given by $C_t = \prod_{\tau=1}^{t-1} D_\tau \tilde{C}_t$ for each $t \in \mathbb{Z}_{++}$. The stochastic consumption growth rate is hence

$$g_t := \frac{C_{t+1}}{C_t} = \frac{\prod_{\tau=1}^{t} D_\tau \tilde{C}_{t+1}}{\prod_{\tau=1}^{t-1} D_\tau \tilde{C}_t} = D_t \tilde{g}_t \tag{22}$$

for any $\{D_t\}_{t=1}^{\infty}$ and for each $t \in \mathbb{Z}_{++}$. Therefore, in the stochastic formulation, the optimal growth rate drops by $1 - \alpha$ upon each disaster, but otherwise the growth rate is characterized exactly as in the deterministic model.

3.2 Balanced growth path

Let us move on to the analysis of the balanced growth path. Throughout this section, we maintain Assumptions 6 and 7. We also presuppose that Assumption 5 is satisfied so that the optimal solution exists. Then it follows from a well-known procedure that the balanced growth path is uniquely determined.

Proposition 3.2. *There is a unique balanced growth path in which output, consumption, and physical and human capital all grow at the same constant rate if and only if inequality* (21) *holds. The associated gross growth rate is given by*

$$\tilde{g}_* = (\tilde{\beta}[\eta + 1 - \delta_H]\zeta_H)^{\frac{1}{1-\theta}} = \left(\beta(\lambda\alpha^\theta + 1 - \lambda)[\eta + 1 - \delta_H]\zeta_H\right)^{\frac{1}{1-\theta}}. \tag{23}$$

The proof is in Appendix A.3. This result suggests that (21) is not only sufficient, but also necessary for the existence of the optimal solution if the solution is to be a balanced growth path.[14] This finding remains valid for any

[14] Along the balanced growth path, we have

$$\textstyle\sum_{t=1}^{\infty} \tilde{\beta}^{t-1} u(\tilde{C}_t) = \sum_{t=1}^{\infty} (\tilde{\beta}\tilde{g}_*^{\theta})^{t-1} u(\tilde{C}_1), \tag{24}$$

production function F as long as Assumptions 6 and 7 are satisfied. Accordingly, we hereafter assume (21) instead of Assumption 5.

Given that the balanced growth rate $\tilde{g}_*$ is expressed as a function of the model's primitives, it is now straightforward to see how different aspects of disasters affect economic growth in the long run. Some simple algebra reveals

$$\frac{\partial \tilde{g}_*}{\partial \zeta_H} = \frac{1}{1-\theta}\frac{\tilde{g}_*}{\zeta_H} > 0, \tag{25}$$

$$\frac{\partial \tilde{g}_*}{\partial \alpha} = \frac{\theta}{1-\theta}\frac{\tilde{g}_*\lambda\alpha^{\theta-1}}{\lambda\alpha^\theta + 1 - \lambda} \gtrless 0 \text{ if } \theta \gtrless 0, \tag{26}$$

$$\frac{\partial \tilde{g}_*}{\partial \lambda} = \frac{1}{1-\theta}\frac{\tilde{g}_*(\alpha^\theta - 1)}{\lambda\alpha^\theta + 1 - \lambda} \lesseqgtr 0 \text{ if } \theta \gtrless 0. \tag{27}$$

Moreover, it is immediately clear from (23) that the balanced growth rate is independent of ζ_K. Therefore, we have proven the following proposition.

Proposition 3.3. *The balanced growth rate $\tilde{g}_*$ is increasing in ζ_H, but is not affected by ζ_K. When $\theta > 0$, the growth rate is increasing in α and decreasing in λ. Conversely, when $\theta < 0$, it is decreasing in α and increasing in λ.*

Several remarks are in order. First, the balanced growth rate is negatively affected by the expected damage $1 - \zeta_H$ of disasters on human capital. Human-targeted disasters decrease the effective productivity in the human capital sector. Because human capital is the long-term growth engine, this diminishes the balanced growth rate. This result intuitively indicates that the balanced growth rate is likely to be smaller in a country where human-related resources are vulnerable to disasters.

Less intuitive is the finding that as long as the damage is restricted to the stock of physical capital, the idyosincratic risk of disasters does not affect the long-run growth rate. One possible implication of this is that protecting human-related resources against disasters is more important than physical capital protection. This is not only from a humanitarian viewpoint, but also because we can sustain a higher growth rate in the long run. We could also view this result as an explanation for recent empirical findings. Our result is based on the optimal solution and hence is only applicable to the case where every resource is efficiently allocated across different sectors. However, suppose, for instance, that physical capital is inefficiently overinvested. Then, because the degradation

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which is finite only if $\tilde{\beta}\tilde{g}_*^\theta < 1$. This inequality coincides with (21) because $\tilde{\beta}\tilde{g}_*^\theta = [(\zeta_H[\eta + 1 - \delta])^\theta(\lambda\alpha^\theta + 1 - \lambda)\beta]^{\frac{1}{1-\theta}}$ and $\theta < 1$.

of physical capital is at the least not harmful, a higher frequency of physically destructive disasters is likely to improve the growth rate.[15] This observation is largely consistent with the suggested explanation for the empirical result of Skidmore and Toya (2002).

The role of aggregate risk is somewhat mixed. Depending on the sign of θ, the risk of disasters can increase or decrease the long-run growth rate. This is primarily because of the consumption-smoothing motive. Recall that a smaller value of θ implies a lower elasticity of intertemporal substitution and hence facilitates savings today if sudden declines in future consumption are expected. This precautionary saving is particularly relevant when the expected decline of future consumption is larger (i.e., when α is smaller and λ is larger). In the face of potential disasters, however, there is another motive that discourages savings today. Saving for future consumption means putting a large amount of resources at risk, which makes it reasonable to consume more today. This makes sense, especially when the expected disasters are highly destructive and the risk is evident. Hence, these distinct motives counteract each other and the relative strength is naturally determined by θ. When $\theta < 0$, for instance, people are not very elastic in terms of intertemporal substitution and, as a result, the consumption-smoothing motive dominates. This is another case in which the long-run growth rate is positively correlated with the magnitude and frequency of disasters.[16]

However, it is important to mention that the comparative statistics in terms of α and λ require careful interpretation. As stated above, the realized growth rate of the original stochastic model will deviate from $\tilde{g}_*$ once a disaster hits the economy. In other words, the *direct* impact of large-scale disasters is not included in $\tilde{g}_*$. The balanced growth rate responds to α and λ only because disasters *might* happen in the future. When an economy-wide disaster does hit

[15]This mechanism only works when the cause of overaccumulation of physical capital is also removed by disasters. Suppose, for instance, that the government introduced a new regulation against some externality (such as noise pollution) but the regulation only applies to newly built factories. In such a case, destruction of old factories, which are exempt from the regulation, will eliminate the cause of overaccumulation.

[16]It may seem strange that a larger magnitude of unpredictable disasters can increase the growth rate, even when λ is close to 1. If $\lambda = 1$, a disaster occurs with certainty in every period, which can be regarded as a form of capital depreciation. In other words, as $\lambda \to 1$, our disaster model converges to the standard growth model where capital stocks depreciate by the amount of $1 - \alpha$. Because capital depreciation always lowers the growth rate in the standard growth model, one could expect that $\tilde{g}_*$ is always increasing in α when λ is sufficiently close to 1. As mentioned earlier, however, $\tilde{g}_*$ is the balanced growth rate in those periods when unpredictable disasters do not hit the economy. When $\lambda = 1$, there is no such thing as a period with no disasters. The observed discontinuity is thus natural.

the economy, the growth rate in the following period will unambiguously decline to $\alpha\tilde{g}_*$. Even though the per-period growth rate will eventually go back to the balanced-growth rate, the average growth rates of those countries which actually experienced a disaster will continue to be lower than otherwise. As a result, if the growth rate of a disaster-stricken country is compared with a counterfactual scenario of no disaster experience, one would always find a significantly negative impact of disaster to growth. This result at least partially explains the recent empirical findings (Raddatz, 2007; Noy, 2009; Hsiang and Jina, 2015).

Aside from the growth rate, we are also interested in how long-run resource allocation is influenced by the risk of disasters. Along the balanced growth path, the raw labor input share $\tilde{n}_*$ in the human capital sector may be written as

$$\tilde{n}_* = (\eta\zeta_H)^{-1}\frac{\tilde{H}_t^*}{\tilde{H}_{t-1}^*} - \eta^{-1}(1-\delta_H) = (\eta\zeta_H)^{-1}\tilde{g}_* - \eta^{-1}(1-\delta_H). \tag{28}$$

The comparative statics on $\tilde{n}_*$ then directly follow from Proposition 3.3, except for ζ_H. This already indicates that the long-run growth rate is adjusted through the reallocation of raw labor. To further investigate this point, let us focus on unpredictable disasters and suppose that the elasticity of intertemporal substitution is relatively small.[17] We know from Proposition 3.3 that a higher frequency of unpredictable disasters will then result in a higher economic growth rate in the long run. What is shown by (28) is that this is achieved by pulling raw labor out of the market and increasing the fraction of time spent developing human capital. In other words, as long as the economy is less elastic in terms of intertemporal substitution, the risk of disasters puts more emphasis on human capital. We can see this most clearly if we look at the ex post composition of physical and human capital $(K/H)^*$, which is defined by

$$(K/H)^* := \frac{\zeta_K K_t^*}{\zeta_H H_t^*} = \frac{\zeta_H[\eta+1-\delta_H]-\tilde{g}_*}{\zeta_H\eta}\tilde{\kappa}_*. \tag{29}$$

Here, $\tilde{\kappa}_*$ is the capital-to-labor ratio in the final goods sector along the balanced growth path, which is implicitly determined by

$$F_k(\tilde{\kappa}_*, 1) = \frac{\zeta_H}{\zeta_K}(\eta+1-\delta_H) - (1-\delta_K). \tag{30}$$

Since $\tilde{\kappa}_*$ is independent of α and λ, this shows that the reaction of capital

[17] The influence of idiosyncratic risk in resource allocation involves more subtleties than one might expect. In Appendix B.2, we provide the comparative statics of each variable in detail.

composition to unpredictable disasters is also parallel to the behavior of the balanced growth rate. This finding is not trivial because, unlike idiosyncratic risks of disasters, the aggregate risk of disasters affect both types of capital in the same manner.

Disasters influence not only the within-period resource allocation, but also the intertemporal resource allocation. A variable that deserves attention in this regard is the savings rate. Along the balanced growth path, the savings rate is given by

$$\tilde{s}_* := 1 - \frac{\tilde{C}_t^*}{\tilde{Y}_t^*} = \frac{\tilde{\kappa}_*}{\zeta_K F(\tilde{\kappa}_*, 1)}[\tilde{g}_* - (1 - \delta_K)]. \tag{31}$$

The comparative statics of $\tilde{s}_*$ and $\tilde{g}_*$ again coincide with respect to α and λ. This indicates that intertemporal consumption reallocation is another major driver behind the disaster-growth relationship. This finding is particularly important when the empirically observed correlation is interpreted. As we have seen, long-run economic growth can be boosted by the risk of disasters. This can be a consequence of the unintended upgrade of otherwise inefficient physical capital or the facilitation of human capital investment. However, these explanations are only part of the story. The positive correlation between growth and disasters can also follow from the precautionary saving, which would not be necessary in the absence of potential disasters. If this is the case, then the high economic growth rate can only be achieved at the cost of suppressed consumption by the present generation.

3.3 Transition phase

In this section, we turn to the economy off the balanced growth path and characterize its behavior. Those interesting properties we have identified in the preceding section have particular relevance if the economy converges to the balanced growth path. Moreover, the transition dynamics are of interest in their own right because the economy may well be in transition to its long-run equilibrium. In the continuous-time setting, the transition dynamics of two-sector endogenous growth models have been extensively studied in the literature (Caballe and Santos, 1993). To the best of our knowledge, however, the behaviors of their discrete-time counterparts are still not well understood. The work of Mitra (1998) is closest to our analysis, but his model assumes the Cobb-Douglas production function. As we will see shortly, assuming the Cobb-Douglas production function would miss out on an important insight regarding an imbalance effect caused by the idiosyncratic (biased) shocks.

To facilitate discussion, we specify the production function to be of the constant elasticity of substitution (CES) form.

Assumption 8. The production function F is given by

$$F(k,l) = \left(\nu k^{\frac{\sigma-1}{\sigma}} + (1-\nu) l^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{32}$$

for some $\nu \in (0,1)$ and $\sigma \in (0,\infty) \setminus \{1\}$ such that

$$\nu^{\frac{\sigma}{\sigma-1}} < \frac{\zeta_H}{\zeta_K}(\eta + 1 - \delta_H) - (1-\delta_K) \text{ if } \sigma > 1 \tag{33}$$

and

$$\nu^{\frac{\sigma}{\sigma-1}} > \frac{\zeta_H}{\zeta_K}(\eta + 1 - \delta_H) - (1-\delta_K) \text{ if } \sigma < 1. \tag{34}$$

We hereafter replace Assumptions 6 and 7 with Assumption 8. The balanced growth path then exists if and only if (21) is satisfied, which we assume throughout this section.

As the next proposition shows, the dynamics of the economy off the balanced growth path is then characterized as a monotonic transition to the long-run equilibrium.

Proposition 3.4. *The optimal path $\{\tilde{g}_t\}_{t=1}^{\infty}$ of the consumption growth rate monotonically converges to the balanced growth rate $\tilde{g}_*$.*

The proof is in Appendix A.4. It is useful to observe that this stability result is shared by the stochastic formulation as well. By (22), the stochastic growth path $\{g_t\}_{t=1}^{\infty}$ is given by

$$g_t = D_t \tilde{g}_t \quad \forall t \in \mathbb{Z}_{++} \tag{35}$$

for any realization $\{D_t\}_{t=1}^{\infty}$ of disaster history. This means that g_t converges to the stochastic balanced growth rate $D\tilde{g}_*$, where D is a random variable following the Bernoulli process. Hence, our characterization of the economy based on the deterministic formulation is justified, not only along the balanced growth path, but also in the transition phase. The convergence itself is not affected by the aggregate risk.

If there is an initial imbalance among the different sectors (i.e, if the initial ratio of capital stocks is not the same as the steady-state ratio), the growth rate in the transition phase will be in general either smaller or grater than the balanced growth rate. In continuous-time settings with the Cobb-Douglass production function, Barro and Sala-i-Martin (1995) provide a local analysis of such an imbalance effect around the steady state and Boucekkine et al. (2008) even present a global analysis based on Gaussian hypergeometric functions. Since our model is discrete-time with a more general production function, we are not able to characterize how exactly the initial state of the economy affects the transition dynamics. But the next proposition says that for two different initial states of the economy, the entire paths of optimal consumption growth rate are unambiguously ranked.

Proposition 3.5. *Consider two different initial states,* (K_0, H_0) *and* (K_0', H_0')*, and let* $\{\tilde{g}_t\}_{t=1}^{\infty}$ *and* $\{\tilde{g}_t'\}_{t=1}^{\infty}$ *be the associated optimal paths of consumption growth rate, respectively. Then* $\tilde{g}_t < \tilde{g}_t'$ *for all* $t \geq 1$ *if and only if* $\tilde{g}_1 < \tilde{g}_1'$*. Similarly,* $\tilde{g}_t > \tilde{g}_t'$ *for all* $t \geq 1$ *if and only if* $\tilde{g}_1 > \tilde{g}_1'$*.*

The proof is in Appendix A.5. This result may be of interest in the context of disasters because, as suggested by Mulligan and Sala-i-Martin (1993), an initial imbalance in capital stocks may be interpreted as a consequence of truly unexpected disasters (i.e., those disasters whose risk is not taken into account in the decision maker's optimization problem). The impact of such disasters simply shift the path of consumption growth rate upwards or downwards, depending on how each of the capital stocks are differentially affected.

What about the idiosyncratic risk of disasters? As it turns out, the potentially biased damage of disasters affects the speed of convergence in a nontrivial manner. We demonstrate this fact by considering two distinct economies with different values of ζ's. Each economy, indexed by $i \in \{A, B\}$, is characterized by the set of parameters $\mathcal{E}^i := (\zeta_K^i, \zeta_H^i; \delta_K, \delta_H, \lambda, \alpha, \theta, \beta, \eta, \nu, \sigma)$. Let $\{\tilde{g}_t^i\}_{t=1}^{\infty}$ be the path of the consumption growth rate of economy i. Our interest is in which of the economies more quickly converges to the balanced growth path when they are different with respect to (ζ_K^i, ζ_H^i).

Since the balanced growth rate (23) depends on ζ_H, each economy, in general, converges to a different balanced growth rate, which we denote by $\tilde{g}_*^i := \lim_{t\to\infty} \tilde{g}_t^i$. To make the comparison meaningful, we restrict our attention to the pair of paths such that $\tilde{g}_1^A/\tilde{g}_*^A = \tilde{g}_1^B/\tilde{g}_*^B$. In other words, we see two paths as comparable as long as the initial states of the economies have the same relative distance to their own balanced growth path. In addition, we say that path

$\{\tilde{g}_t^A\}_{t=1}^\infty$ converges faster than path $\{\tilde{g}_t^B\}_{t=1}^\infty$ if

$$|\tilde{g}_t^A/\tilde{g}_*^A - 1| < |\tilde{g}_t^B/\tilde{g}_*^B - 1| \tag{36}$$

for every $t \geq 2$. We are now ready to state our final proposition.

Proposition 3.6. *Assume $\delta_K = 1$ and let $\{\tilde{g}_t^A\}_{t=1}^\infty$ and $\{\tilde{g}_t^B\}_{t=1}^\infty$ be two comparable paths associated with two distinct economies $\mathcal{E}^A$ and $\mathcal{E}^B$, respectively. Suppose*

$$\zeta_K^A/\zeta_H^A < \zeta_K^B/\zeta_H^B. \tag{37}$$

Then i) $\{\tilde{g}_t^A\}_{t=1}^\infty$ converges faster than $\{\tilde{g}_t^B\}_{t=1}^\infty$ if $\sigma > 1$; ii) $\{\tilde{g}_t^B\}_{t=1}^\infty$ converges faster than $\{\tilde{g}_t^A\}_{t=1}^\infty$ if $\sigma < 1$.

The proof is in Appendix A.6. For this proposition, in order to obtain a clean result, we simply assume that physical capital fully depreciates between periods. The full depreciation assumption is only a good approximation if the time step is taken to be sufficiently long, say a decade or longer. Although unrealistic for a short-term analysis, using such a long time step would be fairly reasonable since our primary interest is in the long-run behavior of the economy.

To fix the context, let us focus on the case where $\zeta_H^A = \zeta_H^B$ so that the two economies converge to the same balanced growth rate. Since these economies are comparable, their initial growth rates are also identical. Then Proposition 3.6 shows that an economy with a larger magnitude of physically destructive disasters converges faster as long as the two inputs in the production function are substitutes. Suppose, for instance, that the growth rate of an economy is initially smaller than the balanced growth rate. In the process of transition to the balanced growth path, this economy can achieve a higher growth rate in every period when they face a higher risk of physically destructive disasters. This result provides yet another explanation for the observed correlation between disaster frequency and the long-run growth rate.

This finding clearly demonstrates the rate-of-return effect of disasters suggested by Skidmore and Toya (2002). Moreover, our result indicates that what also matters is the substitutability between physical capital and effective labor in final goods production. A greater magnitude of physically destructive disasters forces the economy to shift the focus from physical capital to human capital. When the two inputs are substitutes, the productivity decline from the lower level of physical capital is made up for by the accompanying improvement in labor productivity. In this case, reallocating resources from physical to human capital will not significantly affect per-period production. A stronger emphasis on human capital then bumps up the growth rate during the transition phase to the long-run equilibrium. This is not possible when the two inputs are complements. If both inputs are essential for final goods production, the degraded physical capital cannot be easily compensated for by developing human capital, which results in a lower growth rate. It should be worth noting that the imbalance effect identified in Proposition 3.6 would not be observed when the production function is assumed to be of the Cobb-Douglas form, in which the elasticity of substitution is fixed at $\sigma = 1$.

4 Discussion

The analysis in the preceding section, although fairly standard, has some limitations. One might argue, for instance, that our results are driven by the fact that the accumulation of human capital is the sole engine of long-run growth, a key feature of the Lucas model. Some of the clear-cut results, including the independence from physically destructive disasters, would not carry over to other types of growth models. In the AK model with physical and human capital, for instance, the long-run growth rate depends on the equilibrium physical-to-human capital ratio, which in turn is determined by the depreciation rates of both types of capital. What then matters for the rate-of-return effect to the long-run growth rate will be the relative magnitude of the expected damages to each type of capital, just as is the case with our analysis of the transition phase. The precautionary savings effect, on the other hand, will be valid in a wide range of models. In fact, the mechanism behind this effect is well known in the context of risky investment (Levhari and Srinivasan, 1969; Sandomo, 1970; Devereux and Smith, 1994).

Given the importance of θ for the precautionary savings effect, care should be taken with the interpretation of this parameter. It is well known that in the standard expected utility model, θ simultaneously captures two distinct aspects of preference: intertemporal consumption substitution and risk aversion. Depending on which aspect of the preference is represented by θ, empirically plausible values for this parameter will be significantly different (Vissing-Jorgensen and Attanasio, 2003; Bansal and Yaron, 2004, Chen et al., 2013). In our model, what is crucial for the precautionary savings effect is not risk aversion, but elasticity of intertemporal substitution. This can be seen by extending our general framework to the recursive utility model of Epstein and Zin (1989, 1991) and Weil (1990). Appendix B.4 describes the details of such an extension. Applied to the endogenous growth example, the extended model yields the balanced growth rate of the form

$$\tilde{g}_* = \left(\beta(\lambda\alpha^{1-\gamma} + 1 - \lambda)^{\frac{\theta}{1-\gamma}}(\eta + 1 - \delta_H)\zeta_H\right)^{\frac{1}{1-\theta}}, \tag{38}$$

where $\gamma > 0$ is the Arrow-Pratt coefficient of risk aversion and $1/(1-\theta)$ is the elasticity of intertemporal substitution. As shown in Appendix B.4, the comparative statics we presented in Proposition 3.3 all survive. The precautionary savings effect critically depends on θ, not on γ. Empirically, the evidence on the magnitude of θ is mixed. The best available evidence suggests that high and low elasticities of intertemporal substitution are both possible (Hall, 1988; Vissing-Jorgensen, 2002) and there is considerable cross-country heterogeneity in the existing estimates (Havranek, 2015).

The present paper focuses on the social planner's problem with the aim of providing a theoretical benchmark. An obvious next step is to extend the analysis to a decentralized economy where households and firms individually face the risk of disasters. Such an extension may be conducted in line with the incomplete market models like those developed by Aiyagari (1994), Krebs (2003), or Angeletos (2007). In fact, that is what was pursued by Bakkensen and Barrage (2016) in the context of uninsurable cyclone risks. What we call the aggregate risk in our social planner's problem is reinterpreted as the uninsurable investment risk from the perspective of individual households. The impacts of disasters on the aggregate savings and growth in such a decentralized model are similar to our result: the precautionary savings effect due to the uninsured part of the risk and the rate-of-return effect depending on the expected damage of disasters. Although only confirmed in a particular class of growth model, their analysis indicates that what we found in this paper is actually a reasonable benchmark even in a decentralized economy.

5 Conclusion

An increasing number of empirical studies have investigated the economic consequences of disasters. Based on empirical evidence, it has been argued that disasters may have positive impacts on the economy in the long run. Despite its potential importance, little is known about the formal mechanism underlying these empirical observations. In this paper, we attempted to fill this gap by providing a general framework for disaster analysis that can be used for a wide range of economic models. By applying the framework to an endogenous growth model, we demonstrated how standard growth theory can be made consistent with these empirical findings. Likewise, our extensive analysis of the balanced growth path, together with the characterization of the transition phase, provided a number of novel insights.

First, if the damage is restricted to physical capital, disasters will not affect the long-run growth rate as long as the resources are efficiently allocated. If some inefficiency remains in the economy, those disasters that primarily destroy physical capital are likely to boost economic growth. Second, even if no inefficiency is involved, physically destructive disasters can improve the economic growth rate when the economy is still in transition to the long-run equilibrium. This is because the associated change in relative return forces us to put more emphasis on human capital. An important caveat is that reallocating resources from physical to human capital may only achieve a higher growth rate when the two inputs are sufficiently substitutable. Third, the unpredictable nature of disasters plays an important role. Given the risk of disasters, a precautionary policy emerges, depending on the elasticity of intertemporal substitution. Observationally, such a precautionary policy is accompanied by faster human capital accumulation and a higher savings rate.

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A Proofs of propositions

A.1 Proof of Lemma 2.1

Denote by $\pi(D^t)$ the probability of disaster history D^t being realized at period t. To simplify the notation, define

$$\Delta_t := \textstyle\prod_{\tau=1}^t D_\tau \quad \text{and} \quad \tilde{\boldsymbol{y}}_{t+1}(D^t) := \Delta_t^{-1}\boldsymbol{x}_{t+1}(D^t) \tag{39}$$

for each $t \in \mathbb{Z}_+$ and put $\Delta_0 := 1$. Using the homogeneity of the return function and the transition correspondence, we have

$$\begin{aligned}
V(\boldsymbol{x}) &= \sup_{\{\boldsymbol{x}_t\}_{t=1}^\infty} \Bigg\{ R(\boldsymbol{x}, \boldsymbol{x}_1) + \sum_{t=1}^\infty \sum_{D^t} \beta^t R(D_t \boldsymbol{x}_t(D^{t-1}), \boldsymbol{x}_{t+1}(D^t))\pi(D^t) \\
&\qquad\qquad \Bigg| \, \boldsymbol{x}_1 \in \Gamma(\boldsymbol{x}), \boldsymbol{x}_{t+1}(D^t) \in \Gamma(D_t \boldsymbol{x}_t(D^{t-1})), t \in \mathbb{Z}_{++} \Bigg\} \\
&= \sup_{\{\boldsymbol{x}_t\}_{t=1}^\infty} \Bigg\{ R(\boldsymbol{x}, \boldsymbol{x}_1) + \sum_{t=1}^\infty \sum_{D^t} \beta^t \Delta_t^\theta R(\Delta_{t-1}^{-1} \boldsymbol{x}_t(D^{t-1}), \Delta_t^{-1}\boldsymbol{x}_{t+1}(D^t))\pi(D^t) \\
&\qquad\qquad \Bigg| \, \boldsymbol{x}_1 \in \Gamma(\boldsymbol{x}), \Delta_t^{-1}\boldsymbol{x}_{t+1}(D^t) \in \Gamma(\Delta_{t-1}^{-1} \boldsymbol{x}_t(D^{t-1})), t \in \mathbb{Z}_{++} \Bigg\} \\
&= \sup_{\{\tilde{\boldsymbol{y}}_t\}_{t=1}^\infty} \Bigg\{ R(\boldsymbol{x}, \tilde{\boldsymbol{y}}_1) + \sum_{t=1}^\infty \sum_{D^t} \beta^t \Delta_t^\theta R(\tilde{\boldsymbol{y}}_t(D^{t-1}), \tilde{\boldsymbol{y}}_{t+1}(D^t))\pi(D^t) \\
&\qquad\qquad \Bigg| \, \tilde{\boldsymbol{y}}_1 \in \Gamma(\boldsymbol{x}), \tilde{\boldsymbol{y}}_{t+1}(D^t) \in \Gamma(\tilde{\boldsymbol{y}}_t(D^{t-1})), t \in \mathbb{Z}_{++} \Bigg\} \\
&= \sup_{\{\tilde{\boldsymbol{y}}_t\}_{t=1}^\infty} \Bigg\{ R(\boldsymbol{x}, \tilde{\boldsymbol{y}}_1) + \sum_{t=1}^\infty \sum_{D^t} \beta^t \Delta_t^\theta R(\tilde{\boldsymbol{y}}_t, \tilde{\boldsymbol{y}}_{t+1})\pi(D^t) \\
&\qquad\qquad \Bigg| \, \tilde{\boldsymbol{y}}_1 \in \Gamma(\boldsymbol{x}), \tilde{\boldsymbol{y}}_{t+1} \in \Gamma(\tilde{\boldsymbol{y}}_t), t \in \mathbb{Z}_{++} \Bigg\}.
\end{aligned} \tag{40}$$

Notice that in the last equality, we dropped D^{t-1}, the argument of $\tilde{\boldsymbol{y}}_t$. This is possible because a realization of disasters does not affect the feasibility of each modified path $\{\tilde{\boldsymbol{y}}\}_{t=1}^\infty$. Finally, observe

$$\sum_{D^t} \Delta_t^\theta \pi(D^t) = (\lambda\alpha^\theta + 1 - \lambda) \sum_{D^{t-1}} \Delta_{t-1}^\theta \pi(D^{t-1}) = (\lambda\alpha^\theta + 1 - \lambda)^t. \tag{41}$$

Combining (40) and (41) completes the proof.

A.2 Proof of Proposition 3.1

We first show that Assumption 5(a) is satisfied if (21) holds. To see this, suppose $\theta > 0$ and fix $\chi_a > 1$ such that

$$\left(\zeta_H[\eta + 1 - \delta_H]\chi_a\right)^{\theta} \tilde{\beta} < 1, \tag{42}$$

which is possible as long as (21) holds. Let $A := \zeta_H[\eta + 1 - \delta_H] > 0$. By (14), we have $\tilde{H}_t \leq A^t \tilde{H}_0$ for any feasible path and

$$\begin{aligned}
\tilde{K}_t &\leq \tilde{Y}_t + (1-\delta_K)\zeta_K \tilde{K}_{t-1} \\
&= F(\zeta_K \tilde{K}_{t-1}, \zeta_H(1-n_t)\tilde{H}_{t-1}) + (1-\delta_K)\zeta_K \tilde{K}_{t-1} \\
&\leq F(\zeta_K \tilde{K}_{t-1}, A^{t-1}\tilde{H}_0) + (1-\delta_K)\zeta_K \tilde{K}_{t-1} \\
&= A^t \frac{1}{A}\left\{F(\zeta_K \tilde{K}_{t-1}/A^{t-1}, \tilde{H}_0) + (1-\delta_K)\zeta_K \tilde{K}_{t-1}/A^{t-1}\right\},
\end{aligned} \tag{43}$$

where the last equality follows from the homogeneity of F. As in Theorem 1 of Brock and Gale (1969), we rewrite this as

$$\tilde{K}_t/A^t \leq f(\tilde{K}_{t-1}/A^{t-1}; \tilde{H}_0), \tag{44}$$

where the function f is defined by

$$f(x; \tilde{H}_0) := \left\{F(\zeta_K x, \tilde{H}_0) + (1-\delta_K)\zeta_K x\right\}/A. \tag{45}$$

Observe that by Assumptions 6 and 7, $f(\cdot; \tilde{H}_0)$ has a unique fixed point $\bar{x}(\tilde{H}_0) > 0$ and any sequence $(x_t)_{t\geq 0}$ generated by the dynamical system $x_{t+1} = f(x_t; \tilde{H}_0)$ with $x_0 > 0$ converges to the fixed point. Combined with (44), this implies that $\limsup_{t\to\infty} \tilde{K}_t/A^t \leq f(\bar{x}(\tilde{H}_0); \tilde{H}_0) = \bar{x}(\tilde{H}_0)$ and thus

$$\begin{aligned}
\limsup_{t\to\infty} \tilde{C}_t/A^t &\leq \limsup_{t\to\infty} \tilde{Y}_t/A^t \\
&\leq \limsup_{t\to\infty} F(\zeta_K \tilde{K}_{t-1}/A^{t-1}, \tilde{H}_0)/A \\
&\leq F(\zeta_K \bar{x}(\tilde{H}_0), \tilde{H}_0)/A,
\end{aligned} \tag{46}$$

yielding $\limsup_{t\to\infty}(\tilde{C}_t)^{1/t} \leq A$. Hence, for any feasible path, there exists $T_a \in \mathbb{Z}_{++}$ such that

$$\tilde{C}_t < (A\chi_a)^t = \left(\zeta_H[\eta + 1 - \delta_H]\chi_a\right)^t \quad \forall t \geq T_a$$

and therefore

$$\sum_{t=1}^{\infty} \tilde{\beta}^{t-1} \frac{\tilde{C}_t^{\theta}}{\theta} < \sum_{t=1}^{T_a-1} \tilde{\beta}^{t-1} \frac{\tilde{C}_t^{\theta}}{\theta} + \sum_{t=T_a}^{\infty} \tilde{\beta}^{t-1} \frac{(\zeta_H[\eta+1-\delta_H]\chi_a)^{\theta t}}{\theta}, \tag{47}$$

where the right-hand side is finite because χ_a satisfies (42).

We shall show that (21) is sufficient for Assumption 5(b) as well. To see this, suppose $\theta < 0$ and fix $(\tilde{K}_0, \tilde{H}_0) \in \mathbb{R}^2_{++}$ arbitrarily. Choose $\tilde{g}$ such that

$$\tilde{\beta}^{-1/\theta} < \tilde{g} < \zeta_H[\eta+1-\delta_H], \tag{48}$$

which is possible as long as (21) holds. We construct a path feasible from $(\tilde{K}_0, \tilde{H}_0)$ in which there exists $T_b \geq 1$ such that from period T_b onward, $\tilde{C}_t$, $\tilde{Y}_t$, $\tilde{K}_t$, and $\tilde{H}_t$ all grow at the same constant rate $\tilde{g}$. If

$$F(\zeta_K, (\zeta_H[\eta+1-\delta_H]-\tilde{g})\eta^{-1}\tilde{H}_{t-1}/\tilde{K}_{t-1}) + (1-\delta_K)\zeta_K > \tilde{g} \tag{49}$$

for $t = 1$ already, then choosing

$$\tilde{n}_t = \left(\frac{\tilde{g}}{\zeta_H} - (1-\delta_H)\right)\frac{1}{\eta} \tag{50}$$

and

$$\tilde{C}_t = \left(F(\zeta_K, (\zeta_H[\eta+1-\delta_H]-\tilde{g})\eta^{-1}\tilde{H}_{t-1}/\tilde{K}_{t-1}) + (1-\delta_K)\zeta_K - \tilde{g}\right)\tilde{K}_{t-1} \tag{51}$$

for all $t \geq 1$ yields the desired path with $T_b = 1$. If (49) is not satisfied for $t = 1$, fix χ_b such that

$$\frac{\zeta_H[\eta+1-\delta_H]}{\tilde{g}} > \chi_b > 1, \tag{52}$$

which is possible since $\tilde{g}$ satisfies (48). Keep setting $\tilde{n}_t \in (0,1)$ and $\tilde{C}_t \in (0, \tilde{Y}_t)$ such that

$$\frac{F(\zeta_K, (1-\tilde{n}_t)\zeta_H(\tilde{H}_{t-1}/\tilde{K}_{t-1}))(1-n_t)}{F(\zeta_K, (\zeta_H[\eta+1-\delta_H]-\tilde{g})\eta^{-1}(\tilde{H}_{t-1}/\tilde{K}_{t-1}))} < 1, \tag{53}$$

$$\frac{(\eta n_t + 1 - \delta_H)\zeta_H}{\tilde{g}} \geq \chi_b > 1, \tag{54}$$

and $\tilde{C}_t = \tilde{n}_t\tilde{Y}_t$ until (49) is satisfied for some t. This is always possible by setting

n_t sufficiently close to 1. Then, as long as (49) is violated, we have

$$\begin{aligned}\frac{\tilde{H}_t}{\tilde{K}_t} &= \frac{(\eta n_t + 1 - \delta_H)\zeta_H}{F(\zeta_K, (1-n_t)\zeta_H(\tilde{H}_{t-1}/\tilde{K}_{t-1}))(1 - \tilde{C}_t/\tilde{Y}_t) + (1-\delta_K)\zeta_K} \frac{\tilde{H}_{t-1}}{\tilde{K}_{t-1}} \\ &> \frac{(\eta n_t + 1 - \delta_H)\zeta_H}{F(\zeta_K, (\zeta_H[\eta + 1 - \delta_H] - \tilde{g})\eta^{-1}(\tilde{H}_{t-1}/\tilde{K}_{t-1})) + (1-\delta_K)\zeta_K} \frac{\tilde{H}_{t-1}}{\tilde{K}_{t-1}} \\ &\geq \frac{(\eta n_t + 1 - \delta_H)\zeta_H}{\tilde{g}} \frac{\tilde{H}_{t-1}}{\tilde{K}_{t-1}} \\ &\geq \chi_b \frac{\tilde{H}_{t-1}}{\tilde{K}_{t-1}} \\ &\geq \chi_b^t \frac{\tilde{H}_0}{\tilde{K}_0}. \end{aligned} \tag{55}$$

Since

$$\begin{aligned}\lim_{z\to\infty} F(1, z) &= \lim_{k\to 0} F_k(k, l) + \lim_{k\to 0} F_l(k, l)\frac{l}{k} \\ &> \frac{\zeta_H}{\zeta_K}(\eta + 1 - \delta_H) - (1 - \delta_K) \\ &> \tilde{g}/\zeta_K - (1 - \delta_K), \end{aligned} \tag{56}$$

there must exist $T_b > 1$ such that

$$F(\zeta_K, (\zeta_H[\eta + 1 - \delta_H] - \tilde{g})\eta^{-1}\tilde{H}_{T_b-1}/\tilde{K}_{T_b-1}) + (1-\delta_K)\zeta_K > \tilde{g} \tag{57}$$

Hence, choosing $\tilde{n}_t$ and $\tilde{C}_t$ as in (50) and (51), respectively, for all $t \geq T_b$ yields the desired path. The value of objective function associated with this path is then

$$\sum_{t=1}^{T_b-1} \tilde{\beta}^{t-1}\frac{\tilde{C}_t^\theta}{\theta} + \tilde{\beta}^{T_b-1}\sum_{t=0}^{\infty}\tilde{\beta}^t\frac{(\tilde{g}^t\tilde{C}_{T_b})^\theta}{\theta} = \sum_{t=1}^{T_b-1}\tilde{\beta}^{t-1}\frac{\tilde{C}_t^\theta}{\theta} + \tilde{\beta}^{T_b-1}\frac{\tilde{C}_{T_b}^\theta}{\theta(1-\tilde{\beta}\tilde{g}^\theta)} > -\infty, \tag{58}$$

where the inequality follows from (48).

A.3 Proof of Proposition 3.2

Noting that F is homogeneous of degree one, we can write the Euler equations associated with the deterministic problem (19) as

$$\tilde{\beta}(\tilde{g}_t)^{\theta-1}\zeta_K\left(F_k(\tilde{\kappa}_t,1)+1-\delta_K\right)=1, \tag{59}$$

$$\tilde{\beta}(\tilde{g}_t)^{\theta-1}[\eta+1-\delta_H]\zeta_H F_l(\tilde{\kappa}_t,1)=F_l(\tilde{\kappa}_{t-1},1), \tag{60}$$

where $\tilde{g}_t := \tilde{C}_{t+1}/\tilde{C}_t$ is the consumption growth rate and $\tilde{\kappa}_t$ is the capital-to-labor ratio in the final goods sector, which is defined by

$$\tilde{\kappa}_t := \frac{\eta\zeta_K\tilde{K}_t}{[\eta+1-\delta_H]\zeta_H\tilde{H}_t-\tilde{H}_{t+1}}. \tag{61}$$

Solving (59) for $\tilde{g}_t$ yields

$$\tilde{g}_t=\left(\tilde{\beta}\zeta_K[F_k(\tilde{\kappa}_t,1)+1-\delta_K]\right)^{\frac{1}{1-\theta}}, \tag{62}$$

where $\tilde{\kappa}_t$ follows the dynamical system

$$F_k(\tilde{\kappa}_t,1)+1-\delta_K=\frac{\zeta_H}{\zeta_K}(\eta+1-\delta_H)\frac{F_l(\tilde{\kappa}_t,1)}{F_l(\tilde{\kappa}_{t-1},1)}. \tag{63}$$

Denote by $\tilde{\kappa}_*$ the interior steady state of the dynamical system (63), namely, $\tilde{\kappa}_*$ is implicitly defined by

$$F_k(\tilde{\kappa}_*,1)=\frac{\zeta_H}{\zeta_K}(\eta+1-\delta_H)-(1-\delta_K). \tag{64}$$

Under Assumptions 6 and 7, $\tilde{\kappa}_*$ exists and is unique. Let $\tilde{g}_*$ be the associated consumption growth rate, which, by (62) and (64), is expressed as

$$\tilde{g}_*=(\tilde{\beta}[\eta+1-\delta_H]\zeta_H)^{\frac{1}{1-\theta}}=\left(\beta(\lambda\alpha^\theta+1-\lambda)[\eta+1-\delta_H]\zeta_H\right)^{\frac{1}{1-\theta}}. \tag{65}$$

It is clear from (62) that the optimal consumption growth rate $\tilde{g}_t$ is constant if and only if $\tilde{\kappa}_t$ is constant. Since $\tilde{\kappa}_*$ is uniquely determined, this implies that the balanced growth path, if any, must be unique. Let $\tilde{Y}_t^*$,$\tilde{K}_t^*$, $\tilde{H}_t^*$ be the output, physical capital, and human capital along the steady state, respectively. Given $\tilde{Y}_t^* = \zeta_K\tilde{K}_{t-1}^*F(1,1/\tilde{\kappa}_*)$, the growth rates of output and physical capital are identical. It then follows from (11) that the growth rate is constant and equal to $\tilde{g}_*$. Hence, the unique balanced growth path exists if and only if we can choose a positive constant ratio K_t^*/H_t^*, which ensures $\tilde{H}_{t+1}^*/\tilde{H}_t^* = \tilde{g}_*$ at the steady state. We know from (61) that

$$\frac{\tilde{K}_t^*}{\tilde{H}_t^*} = \zeta_K^{-1}\eta^{-1}\left(\zeta_H[\eta + 1 - \delta_H] - \frac{\tilde{H}_{t+1}^*}{\tilde{H}_t^*}\right)\tilde{\kappa}_*, \tag{66}$$

which is positive along the balanced growth path if and only if

$$\zeta_H[\eta + 1 - \delta_H] > \frac{\tilde{H}_{t+1}^*}{\tilde{H}_t^*} = \tilde{g}_* = \left(\beta(\lambda\alpha^\theta + 1 - \lambda)[\eta + 1 - \delta_H]\zeta_H\right)^{\frac{1}{1-\theta}}. \tag{67}$$

This yields the inequality (21).

A.4 Proof of Proposition 3.4

Let $\tilde{g}_t := \tilde{C}_{t+1}/\tilde{C}_t$ be the growth rate of consumption and $\tilde{\kappa}_t$ be the effective capital-to-labor ratio defined by (61). The optimal solution satisfies (62), which means that the behavior of $\tilde{g}_t$ is completely characterized by the dynamics of $\tilde{\kappa}_t$. In particular, since F_k is monotonically decreasing in $\tilde{\kappa}_t$, there is a one-to-one relationship between $\tilde{g}_t$ and $\tilde{\kappa}_t$. Recall that the dynamics of $\tilde{\kappa}_t$ are governed by (63), which may now be written as

$$\nu\left(\frac{F(\tilde{\kappa}_t, 1)}{\tilde{\kappa}_t}\right)^{\frac{1}{\sigma}} + 1 - \delta_K = (\eta + 1 - \delta_H)\frac{\zeta_H}{\zeta_K}\left(\frac{F(\tilde{\kappa}_t, 1)}{F(\tilde{\kappa}_{t-1}, 1)}\right)^{\frac{1}{\sigma}}. \tag{68}$$

Observe that the capital-to-labor ratio $\tilde{\kappa}_*$ at the unique balanced growth path is determined by

$$\nu\left(\frac{F(\tilde{\kappa}_*, 1)}{\tilde{\kappa}_*}\right)^{\frac{1}{\sigma}} + 1 - \delta_K = (\eta + 1 - \delta_H)\frac{\zeta_H}{\zeta_K}. \tag{69}$$

Combining (68) and (69), we have

$$\left(\frac{F(\tilde{\kappa}_t, 1)}{\tilde{\kappa}_t}\right)^{\frac{1}{\sigma}} - \left(\frac{F(\tilde{\kappa}_*, 1)}{\tilde{\kappa}_*}\right)^{\frac{1}{\sigma}} = \frac{\eta + 1 - \delta_H}{\nu}\frac{\zeta_H}{\zeta_K}\left\{\left(\frac{F(\tilde{\kappa}_t, 1)}{F(\tilde{\kappa}_{t-1}, 1)}\right)^{\frac{1}{\sigma}} - 1\right\}. \tag{70}$$

Since $F(\kappa, 1)/\kappa = F(1, 1/\kappa)$ is strictly decreasing in κ, this yields

$$\tilde{\kappa}_t \gtreqless \tilde{\kappa}_* \iff \tilde{\kappa}_t \lesseqgtr \tilde{\kappa}_{t-1}, \tag{71}$$

which in turn implies

$$\tilde{\kappa}_t \gtreqless \tilde{\kappa}_* \implies \tilde{\kappa}_t \gtreqless \tilde{\kappa}_{t+1} \gtreqless \tilde{\kappa}_*. \tag{72}$$

It follows that for any initial value $\tilde{\kappa}_0 > 0$, the path $\{\tilde{\kappa}_t\}_{t=1}^{\infty}$ governed by (68) monotonically converges to $\tilde{\kappa}_*$.

A.5 Proof of Proposition 3.5

Let $\{\tilde{g}_t\}_{t=1}^{\infty}$ and $\{\tilde{g}'_t\}_{t=1}^{\infty}$ be the paths of growth rate associated with the two different initial states, (K_0, H_0) and (K'_0, H'_0), respectively. Also, let $\tilde{\kappa}_t$ and $\tilde{\kappa}'_t$ be the associated effective capital-to-labor ratio defined by (61). Using (70), we have

$$\left(\frac{F(\tilde{\kappa}_t,1)}{\tilde{\kappa}_t}\right)^{\frac{1}{\sigma}} - \left(\frac{F(\tilde{\kappa}'_t,1)}{\tilde{\kappa}'_t}\right)^{\frac{1}{\sigma}} = \frac{\eta+1-\delta_H}{\nu}\frac{\zeta_H}{\zeta_K}\left\{\left(\frac{F(\tilde{\kappa}_t,1)}{F(\tilde{\kappa}_{t-1},1)}\right)^{\frac{1}{\sigma}} - \left(\frac{F(\tilde{\kappa}'_t,1)}{F(\tilde{\kappa}'_{t-1},1)}\right)^{\frac{1}{\sigma}}\right\} \tag{73}$$

for all $t \geq 1$, which implies

$$\begin{aligned}\frac{F(\tilde{\kappa}_t,1)}{\tilde{\kappa}_t} < \frac{F(\tilde{\kappa}'_t,1)}{\tilde{\kappa}'_t} &\iff \frac{F(\tilde{\kappa}_t,1)}{F(\tilde{\kappa}_{t-1},1)} < \frac{F(\tilde{\kappa}'_t,1)}{F(\tilde{\kappa}'_{t-1},1)} \\ &\iff \frac{F(\tilde{\kappa}_t,1)}{F(\tilde{\kappa}'_t,1)} < \frac{F(\tilde{\kappa}_{t-1},1)}{F(\tilde{\kappa}'_{t-1},1)}.\end{aligned} \tag{74}$$

for all $t \geq 1$. Since $F(\kappa,1)/\kappa = F(1,1/\kappa)$ is strictly decreasing in κ, this yields

$$\tilde{\kappa}_t \gtreqless \tilde{\kappa}'_t \iff \tilde{\kappa}_{t-1} \gtreqless \tilde{\kappa}'_{t-1} \tag{75}$$

for all $t \geq 1$. Given (62), it follows that

$$\tilde{g}_t \lesseqgtr \tilde{g}'_t \iff \tilde{g}_1 \lesseqgtr \tilde{g}'_1 \tag{76}$$

as claimed in the proposition.

A.6 Proof of Proposition 3.6

To ease the notation, define

$$\hat{g}^i_t := (\tilde{g}^i_t/\tilde{g}^i_*)^{\sigma(1-\theta)} \tag{77}$$

and

$$\omega^i := \frac{\zeta_K^i}{\zeta_H^i(\eta + 1 - \delta_H)} \tag{78}$$

for each $i \in \{A, B\}$. Observe first that path $\{\tilde{g}_t^A\}_{t=1}^\infty$ converges faster than path $\{\tilde{g}_t^B\}_{t=1}^\infty$ if and only if

$$|\hat{g}_t^A - 1| < |\hat{g}_t^B - 1| \tag{79}$$

for all $t \geq 2$. By the monotonic convergence property in Proposition 3.4, we also have $\lim_{t\to\infty} \hat{g}_t^i = 1$ and

$$\hat{g}_1^i > 1 \iff \hat{g}_t^i > 1 \text{ for all } t \in \mathbb{Z}_{++} \tag{80}$$

for each $i \in \{A, B\}$.

Lemma A.1. *The equation of motion for $\hat{g}_t^i$ is given by*

$$\hat{g}_{t+1}^i = \phi(\hat{g}_t^i; \omega^i) := \left(1 + \nu^\sigma (\omega^i)^{\sigma-1}\left(1 - \frac{1}{(\hat{g}_t^i)^{\frac{\sigma-1}{\sigma}}}\right)\right)^{\frac{\sigma}{\sigma-1}} \tag{81}$$

for all t.

Proof. Observe first that (62) and (63) imply

$$\left(\hat{g}_t^i\right)^{\frac{1}{\sigma}} = \omega^i\left(F_k(\tilde{\kappa}_t^i, 1) + 1 - \delta_K\right) = \nu\omega^i\left(\left(\frac{F(\tilde{\kappa}_t^i, 1)}{\tilde{\kappa}_t^i}\right)^{\frac{1}{\sigma}} + \frac{1 - \delta_K}{\nu}\right) \tag{82}$$

and

$$\omega^i\left(F_k(\tilde{\kappa}_t^i, 1) + 1 - \delta_K\right) = \frac{F_l(\tilde{\kappa}_t^i, 1)}{F_l(\tilde{\kappa}_{t-1}^i, 1)} = \left(\frac{F(\tilde{\kappa}_t^i, 1)}{F(\tilde{\kappa}_{t-1}^i, 1)}\right)^{\frac{1}{\sigma}}. \tag{83}$$

Combining them yields

$$\hat{g}_t^i = \frac{F(\tilde{\kappa}_t^i, 1)}{F(\tilde{\kappa}_{t-1}^i, 1)} \tag{84}$$

and

$$\left(\frac{1 + (1 - \delta_K)/F_k(\tilde{\kappa}_{t+1}^i, 1)}{1 + (1 - \delta_K)/F_k(\tilde{\kappa}_t^i, 1)}\right)^\sigma \hat{g}_t^i = \frac{\tilde{\kappa}_{t+1}^i}{\tilde{\kappa}_t^i} \tag{85}$$

for all $t \geq 1$.

Since $\delta_K = 1$, (82) may be written as

$$\hat{g}_t^i = \left(\nu\omega^i\right)^\sigma F(1, 1/\tilde{\kappa}_t^i) \tag{86}$$

and (85) boils down to $\tilde{\kappa}^i_{t+1} = \hat{g}^i_t \tilde{\kappa}^i_t$, with which (86) implies

$$\hat{g}^i_{t+1} = \left(\nu\omega^i\right)^\sigma F(1, 1/(\hat{g}^i_t \tilde{\kappa}^i_t)). \tag{87}$$

Solving (86) for $\tilde{\kappa}^i_t$ and plugging it into (87) yields (81). □

We are now ready to prove Proposition 3.6. We provide the proof only for the case where $\sigma > 1$ and $\tilde{g}^A_1/\tilde{g}^A_* = \tilde{g}^B_1/\tilde{g}^B_* > 1$. The same argument can be applied to the other cases. Suppose

$$\zeta^A_K/\zeta^A_H < \zeta^B_K/\zeta^B_H \tag{88}$$

so that $\omega^A < \omega^B$. Since $\phi(\hat{g}^i_t; \omega^i)$ in Lemma A.1 is strictly increasing in $\hat{g}^i_t$ and is also strictly increasing in ω^i (provided that $\sigma > 1$ and $\hat{g}^i_t > 1$), it follows that

$$1 < \hat{g}^A_2 = \phi(\hat{g}^A_1; \omega^A) < \phi(\hat{g}^A_1; \omega^B) = \phi(\hat{g}^B_1; \omega^B) = \hat{g}^B_2, \tag{89}$$

which in turn implies

$$1 < \hat{g}^A_3 = \phi(\hat{g}^A_2; \omega^A) < \phi(\hat{g}^A_2; \omega^B) < \phi(\hat{g}^B_2; \omega^B) = \hat{g}^B_3. \tag{90}$$

Continuing in the same fashion for each t, we obtain $1 < \hat{g}^A_t < \hat{g}^B_t$ for all $t \geq 2$, or equivalently, $|\tilde{g}^A_t/\tilde{g}^A_* - 1| < |\tilde{g}^B_t/\tilde{g}^B_* - 1|$ for all $t \geq 2$. Therefore, we conclude that $\{\tilde{g}^A_t\}^\infty_{t=1}$ converges faster than $\{\tilde{g}^B_t\}^\infty_{t=1}$.

B Additional results

B.1 On the case with logarithmic utility

Put $\tilde{\boldsymbol{y}}_t := (K_t, H_t)^\top$ and define

$$\tilde{R}(\tilde{\boldsymbol{y}}_t, \tilde{\boldsymbol{y}}_{t+1}) := \ln(F\left(\zeta_K K_t, \zeta_H H_t - \eta^{-1} H_{t+1}\right) - K_{t+1}). \tag{91}$$

It should then be easy to see that

$$\tilde{R}(D_t \tilde{\boldsymbol{y}}_t(D^{t-1}), \tilde{\boldsymbol{y}}_{t+1}(D^t)) = \tilde{R}(\tilde{\boldsymbol{y}}_t(D^{t-1}), \tilde{\boldsymbol{y}}_{t+1}(D^t)) + \ln(\Delta_t), \tag{92}$$

where Δ_t and $\tilde{\boldsymbol{y}}_t$ are defined by (39). Hence, as in the proof of Lemma 2.1, we can obtain the associated deterministic formulation

$$V(\boldsymbol{x}) = \sup_{\{\tilde{\boldsymbol{y}}_t\}_{t=1}^{\infty}} \left\{ \tilde{R}(\boldsymbol{x}, \tilde{\boldsymbol{y}}_1) + \sum_{t=1}^{\infty} \beta^t \tilde{R}(\tilde{\boldsymbol{y}}_t, \tilde{\boldsymbol{y}}_{t+1}) + v(\alpha, \lambda, \beta) \;\middle|\; \tilde{\boldsymbol{y}}_1 \in \Gamma(\boldsymbol{x}), \tilde{\boldsymbol{y}}_{t+1} \in \Gamma(\tilde{\boldsymbol{y}}_t), t \in \mathbb{Z}_{++} \right\}, \tag{93}$$

where $v(\alpha, \lambda, \beta)$ is a constant defined by

$$v(\alpha, \lambda, \beta) := \sum_{t=1}^{\infty} \sum_{D^t} \beta^t \ln(\Delta_t) \pi(D^t) = \lambda \ln(\alpha) \sum_{t=1}^{\infty} t\beta^t \in \mathbb{R}. \tag{94}$$

A sufficient condition for the objective function in (93) to be bounded on the set of all feasible paths is $\beta < 1$, which is always satisfied.[18] Therefore, we can apply Proposition 2.1 to the case with logarithmic utility.

Observe that the optimal solution of (93) is not influenced by α nor by λ. This means that when the utility function is of a logarithmic form, unpredictable disasters do not affect the growth rate at all. We should emphasize, however, that the welfare implication is unambiguously negative, even in this case. We can see this, in that the constant term $v(\alpha, \lambda, \beta)$ is increasing in α and decreasing in λ. (Be aware that $\ln(\alpha) < 0$ because $\alpha < 1$.)

B.2 More on the comparative statics

Here we report the comparative statics results of $\tilde{\kappa}_*$, $\tilde{n}_*$, and $\tilde{s}_*$. Let us begin with $\tilde{\kappa}_*$, the capital-to-labor ratio employed in final goods production. Since F_k is decreasing by Assumption 6, it is immediately clear from (64) that $\tilde{\kappa}_*$ is increasing in ζ_K and decreasing in ζ_H. This result simply follows from the standard substitution effect. A larger magnitude of disasters makes either physical or human capital relatively scarce, depending on whether the disaster is targeted at physical or human capital. As long as disasters destroy both types of capital in the same proportion, the capital-to-labor ratio is unaffected, and therefore $\tilde{\kappa}_*$ is independent of α and λ.

[18]Notice that with the logarithmic utility, (47) is replaced by

$$\textstyle\sum_{t=T}^{\infty} \beta^{t-1} \ln(\tilde{C}_t) \leq \ln(\zeta_H[\eta + 1 - \delta_H]\chi) \sum_{t=T}^{\infty} t\beta^{t-1}, \tag{95}$$

the right-hand side of which is finite if $\beta < 1$.

As for $\tilde{n}_*$ and $\tilde{s}_*$, using (28) immediately yields

$$\frac{\partial \tilde{n}_*}{\partial \zeta_K} = \frac{1}{\eta \zeta_H} \frac{\partial \tilde{g}_*}{\partial \zeta_K} = 0 \quad \text{and} \quad \frac{\partial \tilde{n}_*}{\partial \zeta_H} \frac{\zeta_H}{\tilde{n}_*} = \frac{\partial \tilde{g}_*}{\partial \zeta_H} \frac{\zeta_H}{\tilde{g}_*} - 1 = \frac{\theta}{1-\theta}. \tag{96}$$

This means that the reaction of $\tilde{n}_*$ to ζ_H will be different from that of $\tilde{g}_*$ if $\theta < 0$. It also follows from (31) that

$$\begin{aligned} \frac{\partial \ln(\tilde{s}_*)}{\partial \zeta_K} &= \left(\frac{1}{\tilde{\kappa}_*} - \frac{F_k(\tilde{\kappa}_*, 1)}{F(\tilde{\kappa}_*, 1)} \right) \frac{\partial \tilde{\kappa}_*}{\partial \zeta_K} - \frac{1}{\zeta_K} \\ &= \frac{1}{\zeta_K} \left(\frac{F_l(\tilde{\kappa}_*, 1)/\varepsilon_*}{F(\tilde{\kappa}_*, 1)} - 1 \right) < (1/\varepsilon_* - 1)/\zeta_K \end{aligned} \tag{97}$$

and

$$\begin{aligned} \frac{\partial \ln(\tilde{s}_*)}{\partial \zeta_H} &= \frac{1}{\tilde{g}_* - (1-\delta_K)} \frac{\partial \tilde{g}_*}{\partial \zeta_H} + \left(\frac{1}{\tilde{\kappa}_*} - \frac{F_k(\tilde{\kappa}_*, 1)}{F(\tilde{\kappa}_*, 1)} \right) \frac{\partial \tilde{\kappa}_*}{\partial \zeta_H} \\ &= \frac{\tilde{g}_*}{\tilde{g}_* - (1-\delta_K)} \frac{1}{(1-\theta)\zeta_H} + \frac{1}{\tilde{\kappa}_*} \frac{\partial \tilde{\kappa}_*}{\partial \zeta_H} \frac{F_l(\tilde{\kappa}_*, 1)}{F(\tilde{\kappa}_*, 1)} \\ &= \frac{1}{\zeta_H} \left(\frac{1}{1-\theta} - \frac{F_l(\tilde{\kappa}_*, 1)/\varepsilon_*}{F(\tilde{\kappa}_*, 1)} \right) > (1/(1-\theta) - 1/\varepsilon_*)/\zeta_H, \end{aligned} \tag{98}$$

where ε_* is the (depreciation-adjusted) capital elasticity of marginal productivity in the balanced growth path, defined by,

$$\varepsilon_* := -\frac{\tilde{\kappa}_* F_{kk}(\tilde{\kappa}_*, 1)}{F_k(\tilde{\kappa}_*, 1)} \frac{\zeta_H(\eta + 1 - \delta_H) - \zeta_K(1-\delta_K)}{\zeta_H(\eta + 1 - \delta_H)}. \tag{99}$$

Therefore,

$$\frac{\partial \tilde{s}_*}{\partial \zeta_K} < 0 \text{ if } \varepsilon_* \geq 1 \quad \text{and} \quad \frac{\partial \tilde{s}_*}{\partial \zeta_H} > 0 \text{ if } \varepsilon_* \geq 1 - \theta. \tag{100}$$

B.3 Continuous-time analogue

Let λ_K be the instantaneous hazard rate of physically destructive small-scale disasters defined by

$$\lambda_K := \lim_{h \to 0} \frac{\Pr(\text{a disaster occurs at a given location during } h \text{ units of time})}{h}. \tag{101}$$

Denote by $\alpha_K \in (0,1)$ the expected fraction of capital which survives each occurrence of this type of disaster. Then, under the assumption of idiosyncratic risk, the aggregate process of physical capital accumulation is

$$K_{t+\Delta t} = \mathbb{E}\left[z_K(\Delta t)\left\{K_t + \left(F(K_t,(1-n_t)H_t) - C_t - \delta_K K_t\right)\Delta t\right\}\right] \tag{102}$$

for sufficiently small $\Delta t > 0$, where $z_K(\Delta t)$ is a random variable such that

$$z_K(\Delta t) = \begin{cases} 1 & \text{with probability } e^{-\lambda_K \Delta t} \\ \alpha_K & \text{with probability } 1 - e^{-\lambda_K \Delta t}. \end{cases} \tag{103}$$

Notice that (102) may be written as

$$\begin{aligned} \frac{K_{t+\Delta t} - K_t}{\Delta t} &= \left(e^{-\lambda_K \Delta t} + \alpha_K(1 - e^{-\lambda_K \Delta t})\right)\left(F(K_t,(1-n_t)H_t) - C_t - \delta_K K_t\right) \\ &\quad + (1-\alpha_K)\frac{e^{-\lambda_K \Delta t} - 1}{\Delta t} K_t. \end{aligned} \tag{104}$$

By taking the limit for $\Delta t \to 0$, we have

$$\dot{K}_t = F(K_t,(1-n_t)H_t) - C_t - \delta_K K_t - (1-\zeta_K)K_t, \tag{105}$$

where we define ζ_K by

$$1 - \zeta_K = \lim_{\Delta t \to 0} \frac{1 - \mathbb{E}[z_K(\Delta t)]}{\Delta t} = \lambda_K(1-\alpha_K). \tag{106}$$

A similar argument yields

$$\dot{H}_t = G(n_t H_t) - \delta_H H_t - (1-\zeta_H)H_t. \tag{107}$$

Hence, the continuous-time analogue of our two-sector endogenous growth model is given by

$$\begin{aligned} V(K,H) = \max \mathbb{E}&\left[\int_0^T e^{-\rho t} u(C_t)dt + e^{-\rho T} V(\alpha K_T, \alpha H_T)\right] \\ \text{s.t. } & \dot{K}_t = F(K_t,(1-n_t)H_t) - C_t - (\delta_K + 1 - \zeta_K)K_t && (108) \\ & \dot{H}_t = G(n_t H_t) - (\delta_H + 1 - \zeta_H)H_t && (109) \\ & H_0 = H, K_0 = K, && (110) \end{aligned}$$

where T is the uncertain timing of economy-wide disaster. The timing is a poisson process with a constant hazard rate $\lambda > 0$. Notice that the assumption of full depreciation of capital is relaxed here. Assuming that the optimal solution of this problem exists, the associated Hamilton-Jacobi-Bellman (HJB) equation is given by

$$\begin{aligned}\rho V(K,H) = \max_{C,n}\Big\{ & u(C) + V_K(K,H)[F(K,(1-n)H) - C - (\delta_K + 1 - \zeta_K)K] \\ & + V_H(K,H)[G(nH) - (\delta_H + 1 - \zeta_H)H] \\ & + \lambda[V(\alpha K, \alpha H) - V(K,H)]\Big\}. \end{aligned} \tag{111}$$

As in Proposition 2.1, one can show that V is homogeneous of degree θ. Therefore, the HJB equation may be written as

$$\begin{aligned}\tilde{\rho} V(K,H) = \max_{C,n}\Big\{ & u(C) + V_K(K,H)[F(K,(1-n)H) - C - \tilde{\delta}_K K] \\ & + V_H(K,H)[G(nH) - \tilde{\delta}_H H]\Big\}, \end{aligned} \tag{112}$$

where

$$\tilde{\rho} := \rho + \lambda - \lambda\alpha^\theta, \quad \tilde{\delta}_i := \delta_i + 1 - \zeta_i \quad i \in \{K, H\}. \tag{113}$$

This coincides with the HJB equation associated with the deterministic version of the problem

$$\begin{aligned} \max \ & \int_0^\infty e^{-\tilde{\rho}t} u(\tilde{C}_t)dt \\ \text{s.t. } & \dot{\tilde{K}}_t = F(\tilde{K}_t, (1-\tilde{n}_t)\tilde{H}_t) - \tilde{C}_t - \tilde{\delta}_K \tilde{K}_t \\ & \dot{\tilde{H}}_t = G(\tilde{n}_t \tilde{H}_t) - \tilde{\delta}_H \tilde{H}_t \\ & \tilde{H}_0 = H, \tilde{K}_0 = K. \end{aligned} \tag{114}$$

Following the standard procedure, the balanced growth rate is computed as

$$\tilde{g}_* = \frac{\eta - (\delta_H + 1 - \zeta_H) - (\rho + \lambda - \lambda\alpha^\theta)}{1-\theta}. \tag{115}$$

A comprehensive characterization of this type of model can be found in Caballe and Santos (1993). The analysis may also be extended to a more general class of two-sector endogenous growth models as demonstrated by Bond et al. (1996).

B.4 Extension to Epstein-Zin-Weil utility

The framework in Section 2 may be extended to the Epstein-Zin-Weil utility model, in which the Bellman equation is given by

$$V(\boldsymbol{x}) = \max_{\boldsymbol{y}\in\Gamma(\boldsymbol{x})} \left\{ R(\boldsymbol{x}, \boldsymbol{y}) + \beta \left(\mathbb{E}\left[(V(D\boldsymbol{y}))^{\frac{1-\gamma}{\theta}} \right] \right)^{\frac{\theta}{1-\gamma}} \right\}, \tag{116}$$

where $\gamma > 0$ is the Arrow-Pratt coefficient of risk aversion.[19] When the two parameters θ and γ happen to satisfy $1-\gamma = \theta$, this problem degenerates into the standard expected-utility framework we relied upon in the main text.

Assume that the solution V of this functional equation exists. Then, as in Proposition 2.1, one can show that V is homogeneous of degree θ under Assumptions 1 and 2. Hence, (116) may be simplified as

$$V(\boldsymbol{x}) = \max_{\tilde{\boldsymbol{y}}\in\Gamma(\boldsymbol{x})} \left\{ R(\boldsymbol{x}, \tilde{\boldsymbol{y}}) + \tilde{\beta} V(\tilde{\boldsymbol{y}}) \right\}, \tag{117}$$

where

$$\tilde{\beta} := \beta \left(\lambda\alpha^{1-\gamma} + 1 - \lambda \right)^{\frac{\theta}{1-\gamma}}. \tag{118}$$

This coincides with the Bellman equation associated with the deterministic version of the problem where the effective discount factor is given by $\tilde{\beta}$. By applying this extended version of the model to the two-sector endogenous growth example in Section 3, we obtain the balanced growth rate

$$\tilde{g}_* = (\tilde{\beta}[\eta + 1 - \delta_H]\zeta_H)^{\frac{1}{1-\theta}} = \left(\beta[\lambda\alpha^{1-\gamma} + 1 - \lambda]^{\frac{\theta}{1-\gamma}} (\eta + 1 - \delta_H)\zeta_H \right)^{\frac{1}{1-\theta}}, \tag{119}$$

which is (38). Some simple algebra reveals

$$\frac{\partial \tilde{g}_*}{\partial \zeta_H} = \frac{1}{1-\theta} \frac{\tilde{g}_*}{\zeta_H} > 0, \tag{120}$$

$$\frac{\partial \tilde{g}_*}{\partial \alpha} = \frac{\theta}{1-\theta} \frac{\tilde{g}_* \lambda \alpha^{-\gamma}}{\lambda\alpha^{1-\gamma} + 1 - \lambda} \gtrless 0 \text{ if } \theta \gtrless 0, \tag{121}$$

$$\frac{\partial \tilde{g}_*}{\partial \lambda} = \frac{\theta}{1-\theta} \frac{\tilde{g}_* \frac{\alpha^{1-\gamma}-1}{1-\gamma}}{\lambda\alpha^{1-\gamma} + 1 - \lambda} \lesseqgtr 0 \text{ if } \theta \gtrless 0, \tag{122}$$

as claimed in the text.

[19] The apparent difference between (116) and the original formulation of Epstein and Zin (1989) is solely a matter of normalization. See, for example, Traeger (2014).