Welfare measures

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1 Basic concepts

Let z be a variable whose level is of interest for individuals, but cannot be controlled by them. For instance, z could be a tax rate or the electricity price. Alternatively, z could be interpreted as the amount of public good supplied by the government. Define the set X of alternatives by

$$X := \mathbb{R}^2_+ := \{(z, m) \in \mathbb{R}^2 \mid z, m \geq 0\}, \tag{1.1}$$

where m is the amount of money. We consider an individual whose preference $\succsim$ is defined on X. To be more precise, we write

$$(z, m) \succ (z', m') \tag{1.2}$$

if she strictly prefers (z, m) to (z', m'). Similarly, we write

$$(z, m) \sim (z', m') \tag{1.3}$$

if she is indifferent between the two alternatives. We then define

$$(z, m) \succsim (z', m') \iff (z, m) \succ (z', m') \text{ or } (z, m) \sim (z', m'), \tag{1.4}$$

meaning that (z, m) is at least as good as (z', m') for her.

Suppose that this individual initially has $(\bar{z}, \bar{m}) \in X$ which we assume is sufficiently large. Consider a public policy that will change the level of z from $\bar{z}$ to $\bar{z}'$. This policy, once implemented, will certainly affect her welfare. What we want to do here is to measure the *welfare impact of this policy in terms of money*. One possible way of doing this is to ask her how much money she would be willing to pay for changing $\bar{z}$ to $\bar{z}'$. In other words, we define the welfare impact of this public policy by the amount CV of money such that

$$(\bar{z}', \bar{m} - \mathrm{CV}) \sim (\bar{z}, \bar{m}). \tag{1.5}$$

This CV is called the *compensating variation*. Notice that CV can be negative if $\bar{z}$ is more preferable than $\bar{z}'$. In that case, $|\mathrm{CV}|$ represents the amount of money this individual would be willing to accept in return for switching to the alternative (less preferable) state.

Another way to measure the same welfare impact in monetary terms is to ask how much money this individual would be willing to accept in return for giving up the proposed policy. Put differently, the welfare impact of this policy could be measured by the amount EV of money such that

$$(\bar{z}, \bar{m} + \mathrm{EV}) \sim (\bar{z}', \bar{m}). \tag{1.6}$$

We often call EV the *equivalent variation*. Again, EV will be negative if $\bar{z}$ is preferred to $\bar{z}'$. In that case, $|\mathrm{EV}|$ represents the amount of money this individual would be willing to pay for preserving the current (more preferable) state.

These two welfare measures do not coincide in general. This is best illustrated by an example. Let us say that the preference $\succsim$ is represented by an (indirect) utility function

$$V(z, m) := z^\alpha m^{1-\alpha} \quad \forall (z, m) \in X \tag{1.7}$$

for some $\alpha \in (0, 1)$, where z is the amount of public good. The compensating variation is then given by

$$\begin{aligned}
(\bar{z}', \bar{m} - \mathrm{CV}) \sim (\bar{z}, \bar{m}) &\iff V(\bar{z}', \bar{m} - \mathrm{CV}) = V(\bar{z}, \bar{m}) \\
&\iff (\bar{z}')^\alpha (\bar{m} - \mathrm{CV})^{1-\alpha} = \bar{z}^\alpha \bar{m}^{1-\alpha} \\
&\iff \mathrm{CV} = \left(1 - \left(\frac{\bar{z}}{\bar{z}'}\right)^{\frac{\alpha}{1-\alpha}}\right) \bar{m}. \qquad (1.8)
\end{aligned}$$

We note that CV is negative if the amount of public good is decreased by this policy (i.e., if $\bar{z}' < \bar{z}$). The equivalent variation, on the other hand, is computed as

$$\begin{aligned}
(\bar{z}, \bar{m} + \mathrm{EV}) \sim (\bar{z}', \bar{m}) &\iff V(\bar{z}, \bar{m} + \mathrm{EV}) = V(\bar{z}', \bar{m}) \\
&\iff \bar{z}^\alpha (\bar{m} + \mathrm{EV})^{1-\alpha} = (\bar{z}')^\alpha \bar{m}^{1-\alpha} \\
&\iff \mathrm{EV} = \left(\left(\frac{\bar{z}'}{\bar{z}}\right)^{\frac{\alpha}{1-\alpha}} - 1\right) \bar{m}. \qquad (1.9)
\end{aligned}$$

It should be easy to see that $\mathrm{CV} \neq \mathrm{EV}$.[1]

Then which one should we use if we want to appropriately measure the welfare impact of public policies? The answer depends on how we define the baseline against which the welfare measure is constructed. In computing the compensating variation, we take the status quo $\bar{z}$ as the baseline. By using CV as a welfare measure, we implicitly assume that individuals should pay the fair cost (or receive the fair benefit) if the situation is to be changed to an alternative state $\bar{z}'$. This suggests that CV should be a relevant welfare measure if

[1] Notice that $\lim_{\bar{z}' \to \bar{z}} (\mathrm{CV} - \mathrm{EV})/\mathrm{EV} = 0$, which implies that the difference is negligible when the change made by the policy is marginal. Also, there is one important case where the two welfare measures always coincide — the case of quasilinear utility function. Assume that the preference is represented by a quasilinear utility function $V(z, m) := v(z) + m$ and verify that $\mathrm{CV} = \mathrm{EV}$ in this case.

the current situation can be seen as legitimate. Consider, for instance, a public policy which limits the height of buildings so as not to cause an unreasonable obstruction of preexisting views or sunlight. If we think that city developers have a legitimate right to build as high buildings as they want, then CV would be the preferable choice in measuring the welfare impact induced by the height restriction. The equivalent variation, on the other hand, takes the alternative state $\bar{z}'$ as the baseline, indicating that people should receive the fair benefit (or pay the fair cost) if we stick to the current state. Hence, it would be preferable to use EV when the alternative state is seen as more legitimate than the current one. In the case of height-restriction example, one could argue that city residents have a legitimate right to enjoy sunlight. If this is the case, then EV would be the appropriate welfare measure.

2 Measuring welfare changes: two approaches

In theory, the two welfare measures we discussed above appropriately capture the welfare changes made by implementation of public policies. In practice, however, it is not easy to actually compute their values. Computing CV or EV requires the information about preference. But the preference is private information which the government is unable to access in many cases. How can we measure the policy-induced welfare changes then?

Roughly speaking, there are two approaches. One way to go is to directly or indirectly ask people to tell us about their preference. This type of approach is often called *stated preference methods*. In the stated preference method approach, we pose to respondents a series of hypothetical questions involving trade-offs between z and m. From the responses, we can then infer the monetary value of welfare impact induced by changes in z. One might think that this approach is a bit naive and is based on a presumption that one could elicit the necessary information by simply asking questions. After all, there is no guarantee that people respond to the hypothetical questions in a reliable and consistent manner. In fact, the validity of stated preference methods has been questioned by economists for a long time.[2] Over the past two decades, however, the methodology has been significantly developed and now the approach is widely used in many of the applied fields. Those who are interested in this approach would find Haab and McConnel (2002) as a good introduction to the literature.

An alternative and more conventional approach is called *revealed preference methods*. In this approach, we turn to the market where people's actual (as opposed to hypothetical) choices can be observed. By observing people's behavior in the market, we can infer how much value they attach to each of the possible alternatives. In other words, preference is revealed in the market. Of course, in many cases, those goods and services affected by public policies are not traded in the market. Nowhere can we find a market where beautiful scenery or clean air is traded. Preference about these goods and services is

[2] See Hausman (2012) for a recent discussion.

therefore not directly observable. If there is a systematic relationship between unobservable and observable aspects of preference, however, we may recover the unobservable part from the observable one. For example, the quality of a natural park is not traded in the market. But one might be able to infer the value of natural parks by observing how much money people are spending in order to visit the parks. In this case, we exploit the systematic relationship between the perceived value of parks (unovservable) and people's decision to visit there (observable).

In what follows, we put our focus on the latter approach and briefly cover its theoretical background.

3 Price-changing policy

We first consider a public policy which affects the price of some good traded in the market. Imagine, for instance, that the government changes its energy policy, shifting emphasis from fossil fuel to renewables. Such a policy will eventually change the electricity price, which in turn affects people's welfare. How can we measure the welfare impact associated with this policy in monetary terms?

Denote by p the electricity price and let us say that the policy induces a price change from $\bar{p}$ to $\bar{p}'$. For simplicity, we divide the entire goods traded in the economy into two broad categories: electricity-related goods and the rest. Then consumer's utility maximization problem is given by

$$\max_{x,y} U(x,y) \ \text{ s.t. } (x,y) \in B(p,m) := \{(x,y) \in \mathbb{R}^2_+ \mid px + y = m\}, \tag{3.1}$$

where x is the amount of electricity-related goods and y is the amount of other goods (which is taken to be the numéraire). Denote by $x^d(p,m), y^d(p,m)$ the demand functions, namely,

$$(x^d(p,m), y^d(p,m)) \in \operatorname*{arg\,max}_{(x,y)\in B(p,m)} U(x,y) \quad \forall (p,m). \tag{3.2}$$

Then we define the indirect utility function by

$$V(p,m) := U(x^d(p,m), y^d(p,m)) \quad \forall (p,m). \tag{3.3}$$

By definition, the compensating variation is CV such that

$$V(\bar{p}', \bar{m} - \text{CV}) = V(\bar{p}, \bar{m}) \tag{3.4}$$

while the equivalent variation is EV such that

$$V(\bar{p}, \bar{m} + \text{EV}) = V(\bar{p}', \bar{m}). \tag{3.5}$$

We want to somehow compute CV and EV based on the observation of people's behavior in the market.

3.1 Expenditure function

In order to characterize CV and EV in a more tractable form, it is useful to first consider the so-called *expenditure minimization problem*. In our context, the expenditure minimization problem is formulated as

$$\min_{x,y} \; px + y \quad \text{s.t.} \; (x, y) \in I(u) := \{(x, y) \in \mathbb{R}^2_+ \mid U(x, y) = u\} \tag{3.6}$$

for each $u \in \mathbb{R}$. Denote by $x^c(p, u), y^c(p, u)$ the *compensated demand functions*, namely,

$$(x^c(p, u), y^c(p, u)) \in \arg\min_{(x,y) \in I(u)} \{px + y\} \quad \forall (p, u). \tag{3.7}$$

Then we define the *expenditure function* by

$$e(p, u) := px^c(p, u) + y^c(p, u) \quad \forall (p, u). \tag{3.8}$$

As one might expect, there is a close relationship between utility maximization and expenditure minimization.

> Theorem 3.1: **Demands, indirect utility, and expenditure**
>
> The following result, which we will use repeatedly in this course, characterizes this relationship in a handy manner. For any p and m,
>
> (a) $x^c(p, V(p, m)) = x^d(p, m)$,
>
> (b) $e(p, V(p, m)) = m$.
>
> Moreover,
>
> (c) $x^d(p, e(p, u)) = x^c(p, u)$,
>
> (d) $V(p, e(p, u)) = u$
>
> for all p and u.

Proof. Left as an exercise. ■

Another useful result, which you might be already familiar with, is *Shephard's lemma*. Shephard's lemma simply says that we can derive the compensated demand function from expenditure function. The following theorem formally states this result.

> Theorem 3.2: **Shephard's lemma**
>
> For any p and u,
>
> $$\frac{\partial e(p, u)}{\partial p} = x^c(p, u). \tag{3.9}$$

Proof. Left as an exercise. ■

We could interpret the left-hand side of (3.9), $\partial e / \partial p$, as the *marginal willingness to pay* for price-changing policy. It captures how much money you would be able to save without sacrificing the current level of utility when the price declines marginally. As Theorem 3.2 states, the marginal willingness to pay is equal to the current level of consumption. This result should not be a surprise. If the unit price p marginally decreases, say by 1 cent, then your preferred consumption bundle would barely change and, as a result, you would be able to save $1 \times x$ cent while keeping the utility level unchanged.

The difficulty in computing CV and EV lies in the fact that they are defined by the indirect utility function as in (3.4) and (3.5). The indirect utility function is not observable because it is purely a theoretical construct. We need some formula which connects CV and EV with something that we can observe. To this end, the next theorem provides a promising first step.

Theorem 3.3: **Compensated-demand representation of CV and EV**

Consider a public policy which changes the price from $\bar{p}$ to $\bar{p}'$. The compensating variation associated with this policy may be computed by

$$\mathrm{CV} = \int_{\bar{p}'}^{\bar{p}} \frac{\partial e(p,\bar{u})}{\partial p} dp = \int_{\bar{p}'}^{\bar{p}} x^c(p,\bar{u}) dp, \tag{3.10}$$

where $\bar{u} := V(\bar{p},\bar{m})$. Similarly, we can compute the equivalent variation by

$$\mathrm{EV} = \int_{\bar{p}'}^{\bar{p}} \frac{\partial e(p,\bar{u}')}{\partial p} dp = \int_{\bar{p}'}^{\bar{p}} x^c(p,\bar{u}') dp, \tag{3.11}$$

where $\bar{u}' := V(\bar{p}',\bar{m})$.

Proof. Putting $\bar{m}' := \bar{m} - \mathrm{CV}$, we have

$$\begin{aligned} V(\bar{p}',\bar{m}') = V(\bar{p},\bar{m}) \implies & \begin{array}{l} \bar{m} = e(\bar{p}, V(\bar{p},\bar{m})) \\ \bar{m}' = e(\bar{p}', V(\bar{p}',\bar{m}')) = e(\bar{p}', V(\bar{p},\bar{m})) \end{array} \\ \implies & \mathrm{CV} = \bar{m} - \bar{m}' = e(\bar{p},\bar{u}) - e(\bar{p}',\bar{u}), \end{aligned} \tag{3.12}$$

where $\bar{u} := V(\bar{p},\bar{m})$. Hence, CV may be computed as

$$\mathrm{CV} = e(\bar{p},\bar{u}) - e(\bar{p}',\bar{u}) = \int_{\bar{p}'}^{\bar{p}} \frac{\partial e(p,\bar{u})}{\partial p} dp = \int_{\bar{p}'}^{\bar{p}} x^c(p,\bar{u}) dp. \tag{3.13}$$

Similarly, put $\bar{m}'' := \bar{m} + \mathrm{EV}$ and observe

$$\begin{aligned} V(\bar{p},\bar{m}'') = V(\bar{p}',\bar{m}) \implies & \begin{array}{l} \bar{m} = e(\bar{p}', V(\bar{p}',\bar{m})) \\ \bar{m}'' = e(\bar{p}, V(\bar{p},\bar{m}'')) = e(\bar{p}, V(\bar{p}',m)) \end{array} \\ \implies & \mathrm{EV} = \bar{m}'' - \bar{m} = e(\bar{p},\bar{u}') - e(\bar{p}',\bar{u}'), \end{aligned} \tag{3.14}$$

where $\bar{u}' := V(p',m)$. Hence, EV may be computed as

$$\mathrm{EV} = e(\bar{p},\bar{u}') - e(\bar{p}',\bar{u}') = \int_{\bar{p}'}^{\bar{p}} \frac{\partial e(p,\bar{u}')}{\partial p} dp = \int_{\bar{p}'}^{\bar{p}} x^c(p,\bar{u}') dp, \tag{3.15}$$

as claimed. ■

Theorem 3.3 states that we can compute CV and EV if we have the information about compensated demand function of consumers. As an immediate corollary, we obtain the next theorem, which compares CV with EV in terms of their magnitude.

Theorem 3.4: Comparing CV and EV
If x is a normal good, then
$$\text{CV} \leq \text{EV} \tag{3.16}$$
for any price-changing policy.

Proof. Since x is a normal good (i.e., $x^d(p,m)$ is increasing in m),
$$\frac{\partial x^c(p,u)}{\partial u} = \frac{\partial x^d(p,m)}{\partial m}\bigg|_{m=e(p,u)} \frac{\partial e(p,u)}{\partial u} \geq 0 \quad \forall (p,u), \tag{3.17}$$
which means that $x^c(p,u)$ is increasing in u. There are two cases to consider:

(a) If $\bar{p}' > \bar{p}$, then
$$\bar{u} := V(\bar{p},\bar{m}) > V(\bar{p}',\bar{m}) =: \bar{u}' \tag{3.18}$$
and hence
$$x^c(p,\bar{u}) \geq x^c(p,\bar{u}') \quad \forall p. \tag{3.19}$$
We thus have
$$\bar{p}' > \bar{p} \implies \text{CV} = -\int_{\bar{p}}^{\bar{p}'} x^c(p,\bar{u})dp \leq -\int_{\bar{p}}^{\bar{p}'} x^c(p,\bar{u}')dp = \text{EV}. \tag{3.20}$$

(b) If $\bar{p}' < \bar{p}$, on the other hand,
$$\bar{u} := V(\bar{p},\bar{m}) < V(\bar{p}',\bar{m}) =: \bar{u}' \tag{3.21}$$
and hence
$$x^c(p,\bar{u}) \leq x^c(p,\bar{u}') \quad \forall p. \tag{3.22}$$
Therefore,
$$\bar{p}' < \bar{p} \implies \text{CV} = \int_{\bar{p}'}^{\bar{p}} x^c(p,\bar{u})dp \leq \int_{\bar{p}'}^{\bar{p}} x^c(p,\bar{u}')dp = \text{EV}. \tag{3.23}$$

Combining (3.20) and (3.23) proves the claim of the theorem. ∎

We have so far characterized the two alternative welfare measures in terms of x^c, the compensated demand function. Notice that we are still not able to compute CV and EV from data because the compensated demand function is not observable in general. Fortunately, there is a systematic relationship between these welfare measures and people's behavior in the market. More precisely, both CV and EV are related to *consumer's surplus*. Unlike compensating variation and equivalent variation, consumer's surplus can be computed from data (i.e., demand function). Therefore, it is hoped, we may be able to somehow characterize CV and EV in terms of observable information if we can clarify how these measures are related to consumer's surplus. This is the task we will turn to next.

3.2 Consumer's surplus

Recall that *consumer's surplus* associated with the consumption of x at the price of $\bar{p}$ is defined by

$$\mathrm{CS} := \int_{\bar{p}}^{\infty} x^d(p, \bar{m})dp. \tag{3.24}$$

We learned in the introductory microeconomics course that consumer's surplus *approximately* measures the consumer's welfare. If we use CS as an approximate measure of welfare, the welfare impact of price-changing policy is given by

$$\int_{\bar{p}'}^{\infty} x^d(p, \bar{m})dp - \int_{\bar{p}}^{\infty} x^d(p, \bar{m})dp = \int_{\bar{p}'}^{\bar{p}} x^d(p, \bar{m})dp =: \Delta\mathrm{CS} \tag{3.25}$$

The natural question to ask is how ΔCS is related to CV and EV. The following theorem clarifies the relationship between the three welfare measures.

Theorem 3.5: **Consumer's surplus as an approximate welfare measure**
Suppose that x is a normal good. Then

$$\mathrm{CV} \leq \Delta\mathrm{CS} \leq \mathrm{EV} \tag{3.26}$$

for any price-changing policy. Moreover, if the income elasticity of demand is constant, the approximation error is given by

$$\left| \frac{\mathrm{CV} - \Delta\mathrm{CS}}{\Delta\mathrm{CS}} \right| \approx \eta \frac{|\Delta\mathrm{CS}|}{2\bar{m}} \approx \left| \frac{\mathrm{EV} - \Delta\mathrm{CS}}{\Delta\mathrm{CS}} \right|, \tag{3.27}$$

where η is the constant income elasticity.

Proof. Notice that $V(p, \bar{m}) > V(\bar{p}, \bar{m})$ for any $p < \bar{p}$ because a lower price is always more preferable. Since x is a normal good, we then obtain

$$x^d(p, \bar{m}) = x^c(p, V(p, \bar{m})) \geq x^c(p, V(\bar{p}, \bar{m})) = x^c(p, \bar{u}) \quad \forall p < \bar{p}, \tag{3.28}$$

which implies that

$$\bar{p}' < \bar{p} \implies \Delta\mathrm{CS} = \int_{\bar{p}'}^{\bar{p}} x^d(p, \bar{m})dp \geq \int_{\bar{p}'}^{\bar{p}} x^c(p, \bar{u})dp = \mathrm{CV}. \tag{3.29}$$

For any $p > \bar{p}$, on the other hand, we have $V(p, \bar{m}) < V(\bar{p}, \bar{m})$ and hence

$$x^d(p, \bar{m}) = x^c(p, V(p, \bar{m})) \leq x^c(p, V(\bar{p}, \bar{m})) = x^c(p, \bar{u}) \quad \forall p > \bar{p}. \tag{3.30}$$

It then follows that

$$\bar{p}' > \bar{p} \implies \Delta\mathrm{CS} = -\int_{\bar{p}}^{\bar{p}'} x^d(p, \bar{m})dp \geq -\int_{\bar{p}}^{\bar{p}'} x^c(p, \bar{u})dp = \mathrm{CV}. \tag{3.31}$$

Combining (3.29) and (3.31) yields

$$\Delta\text{CS} \geq \text{CV}, \tag{3.32}$$

as required. Similarly, observe that for any $p > \bar{p}'$, it should be the case that $V(p, \bar{m}) < V(\bar{p}', \bar{m})$. Since x is a normal good, then

$$x^d(p, \bar{m}) = x^c(p, V(p, \bar{m})) \leq x^c(p, V(\bar{p}', \bar{m})) = x^c(p, \bar{u}') \quad \forall p > \bar{p}', \tag{3.33}$$

which implies that

$$\bar{p} > \bar{p}' \implies \Delta\text{CS} = \int_{\bar{p}'}^{\bar{p}} x^d(p, \bar{m}) dp \leq \int_{\bar{p}'}^{\bar{p}} x^c(p, \bar{u}') dp = \text{EV}. \tag{3.34}$$

For any $p < \bar{p}'$, on the other hand, we have $V(p, \bar{m}) > V(\bar{p}', \bar{m})$ and hence

$$x^d(p, \bar{m}) = x^c(p, V(p, \bar{m})) \geq x^c(p, V(\bar{p}', \bar{m})) = x^c(p, \bar{u}') \quad \forall p < \bar{p}'. \tag{3.35}$$

It then follows that

$$\bar{p} < \bar{p}' \implies \Delta\text{CS} = -\int_{\bar{p}}^{\bar{p}'} x^d(p, \bar{m}) dp \leq -\int_{\bar{p}}^{\bar{p}'} x^c(p, \bar{u}) dp = \text{EV}. \tag{3.36}$$

Combining (3.34) and (3.36) yields

$$\Delta\text{CS} \leq \text{EV}, \tag{3.37}$$

as desired. This completes the first half of the proof.

To prove the latter half of the theorem, suppose that the income elasticity of demand is constant, namely, there exists $\eta \in \mathbb{R}_+$ such that

$$\eta = \frac{\partial x^d(p, m)}{\partial m} \frac{m}{x^d(p, m)} \quad \forall (p, m). \tag{3.38}$$

Notice that this is a first-order differential equation in terms of m. Integrating both sides of the equation over m yields

$$x^d(\bar{p}, m) = x^d(\bar{p}, \bar{m}) \left(\frac{m}{\bar{m}}\right)^\eta \quad \forall m. \tag{3.39}$$

Hence,

$$\frac{\partial e(p, u)}{\partial p} = x^c(p, u) = x^d(p, e(p, u)) = x^d(p, \bar{m}) \left(\frac{e(p, u)}{\bar{m}}\right)^\eta \quad \forall (p, u) \tag{3.40}$$

or

$$\frac{1}{1-\eta} \frac{\partial (e(p, u))^{1-\eta}}{\partial p} = \bar{m}^{-\eta} x^d(p, \bar{m}) \quad \forall (p, u), \tag{3.41}$$

which is a first-order differential equation in terms of p. Integrate the both sides over p and obtain

$$\begin{aligned} \bar{m}^{-\eta} \underbrace{\int_{\bar{p}'}^{\bar{p}} x^d(p, \bar{m}) dp}_{=\Delta\text{CS}} &= \frac{1}{1-\eta} \int_{\bar{p}'}^{\bar{p}} \frac{\partial (e(p, u))^{1-\eta}}{\partial p} dp \\ &= \frac{1}{1-\eta} \left((e(\bar{p}, u))^{1-\eta} - (e(\bar{p}', u))^{1-\eta} \right) \quad \forall u. \end{aligned} \tag{3.42}$$

In particular, setting $u = \bar{u} := V(\bar{p}, \bar{m})$ in (3.42) yields

$$\begin{aligned} e(\bar{p}', \bar{u}) &= \left(\bar{m}^{1-\eta} - \frac{1-\eta}{\bar{m}^{\eta}} \Delta \text{CS} \right)^{\frac{1}{1-\eta}} \\ &= \bar{m} \left(1 - \frac{1-\eta}{\bar{m}} \Delta \text{CS} \right)^{\frac{1}{1-\eta}} \\ &\approx \bar{m} - \Delta \text{CS} + \frac{\eta}{2\bar{m}} \left(\Delta \text{CS} \right)^2, \end{aligned} \tag{3.43}$$

where the third line uses the Taylor approximation. The approximation is fairly good if the term $\frac{1-\eta}{\bar{m}} \Delta \text{CS}$ is sufficiently close to zero. It then follows that

$$\text{CV} = \bar{m} - e(\bar{p}', \bar{u}) \approx \Delta \text{CS} - \frac{\eta}{2\bar{m}} \left(\Delta \text{CS} \right)^2. \tag{3.44}$$

Similarly, setting $u = \bar{u}' := V(\bar{p}', \bar{m})$ in (3.42) yields

$$\begin{aligned} e(\bar{p}, \bar{u}') &= \left(\bar{m}^{1-\eta} + \frac{1-\eta}{\bar{m}^{\eta}} \Delta \text{CS} \right)^{\frac{1}{1-\eta}} \\ &= \bar{m} \left(1 + \frac{1-\eta}{\bar{m}} \Delta \text{CS} \right)^{\frac{1}{1-\eta}} \\ &\approx \bar{m} + \Delta \text{CS} + \frac{\eta}{2\bar{m}} \left(\Delta \text{CS} \right)^2 \end{aligned} \tag{3.45}$$

and

$$\text{EV} = e(\bar{p}, \bar{u}') - \bar{m} \approx \Delta \text{CS} + \frac{\eta}{2\bar{m}} \left(\Delta \text{CS} \right)^2. \tag{3.46}$$

The expression (3.27) immediately follows from (3.44) and (3.46). ■

This theorem, which was originally proven by Willig (1976), has an important policy implication. First, (3.26) shows that in general, consumer's surplus is not equivalent to compensating variation nor equivalent variation. In other words, consumer's surplus is not the exact welfare measure. To be more precise, ΔCS always lies in between CV and EV.[3] It is in this sense that consumer's surplus approximates the welfare impact induced by policy intervention. Then the question is how precise the approximation will be. One answer to this question is provided by (3.27), which suggests that the approximation error is given by $\eta \frac{|\Delta \text{CS}|}{2\bar{m}}$. Hence, if the demand for x is inelastic to changes in income (i.e., if η is sufficiently small), and if ΔCS is sufficiently small relative to the baseline income $\bar{m}$, then ΔCS can be a good approximation of the welfare impact. In particular, this condition is likely to be met when the good in question is mainly consumed for recreational purposes. This result therefore provides a theoretically rigorous basis for the use of consumer's surplus in applied studies.

[3] This implies that ΔCS provides an upper bound and a lower bound of CV and EV, respectively.

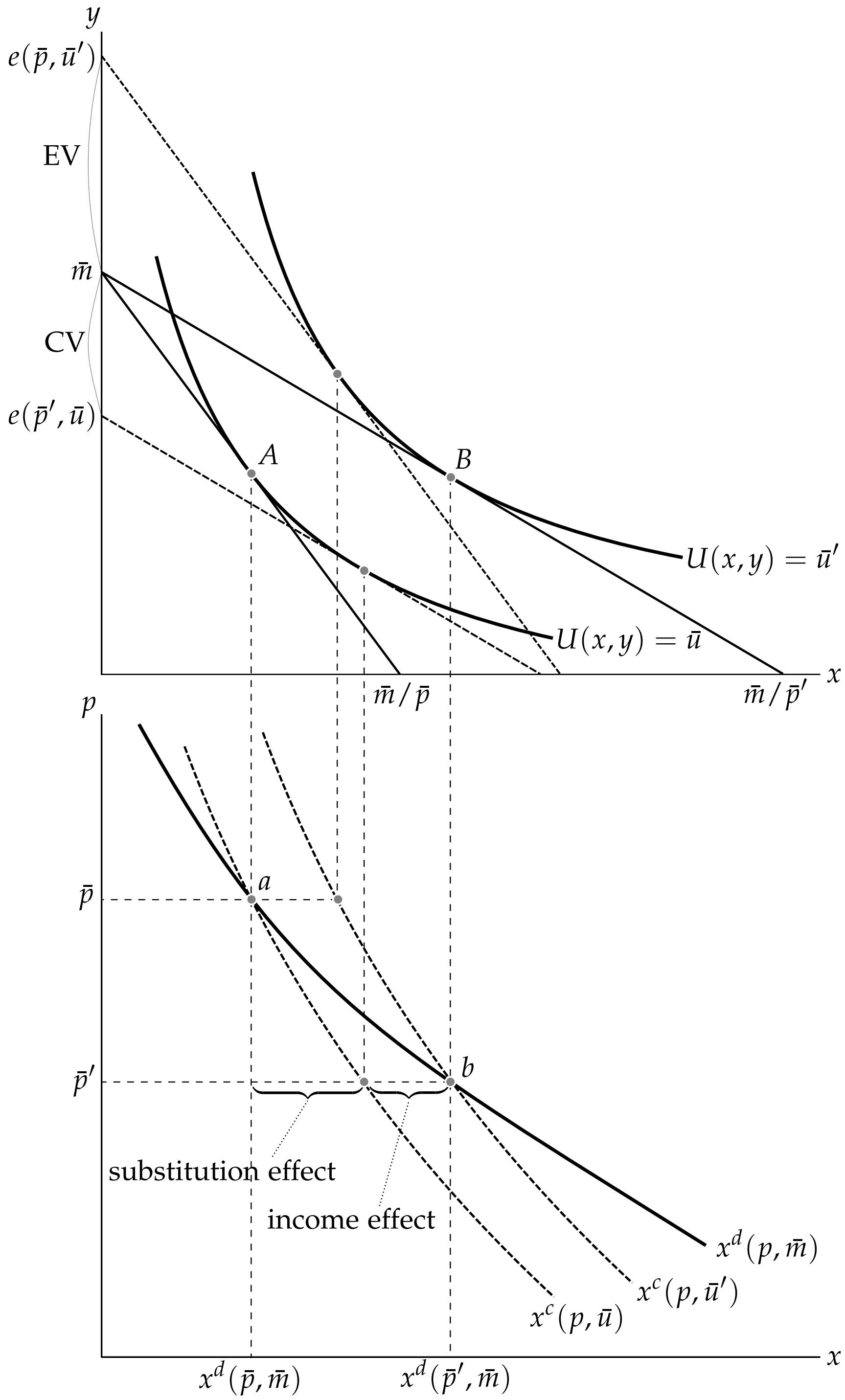

Figure 1: CV and EV for price-changing policies

4 Quantity-changing policy

The discussion in the preceding section clarifies how we can (approximately) measure the welfare impact caused by price-changing policies. Both compensating variation and equivalent variation in response to a price change can be roughly measured by the associated consumer's surplus, especially when the income effect is fairly small. In many cases of interest, however, public policies do not only change the price of market goods, but also involves changes in *quantity* or *quality* of public goods. Consider, for instance, a renovation project of an old city district. Such a project, once successfully implemented, will affect people's welfare by improving the benefit people will gain from visiting the district. For another example, imagine a situation where the national government is planing to set up a new TV channel. People will benefit from this plan because a wider variety of TV shows would then become available. Aside from these examples, one could think of more classic examples such as air-quality improvement through pollution regulations or public provision of high-quality drinkable water.

To see how we can construct an appropriate welfare measure for these quantity-changing policies, denote by q the quantity or quality of a particular public good. For simplicity, we assume that there exists at least one market good whose consumption is somehow related to q. If q is the quality of drinkable water supplied by the government, for instance, it is quite likely that the demand for bottled water or water filters will change depending on q. If q represents the quality of an old city district, on the other hand, the frequency of visit (or the number of visitors) to the district will be more or less affected by q.

As in the preceding section, we divide the entire goods traded in the economy into two broad categories: one related to q and one not related. Then consumer's utility maximization problem is formulated as

$$\max_{x,y} U(x,y,q) \quad \text{s.t. } (x,y) \in B(p,m) := \{(x,y) \in \mathbb{R}^2_+ \mid px + y = m\}, \tag{4.1}$$

where x is the amount of goods whose consumption is related to the public good and y is the amount of other goods (which is taken to be the numéraire). Notice that the level q of public good cannot be chosen by consumers. Denote by $x^d(p,q,m), y^d(p,q,m)$ the demand functions, namely,

$$(x^d(p,q,m), y^d(p,q,m)) \in \underset{(x,y)\in B(p,m)}{\arg\max} \; U(x,y,q) \quad \forall (p,q,m). \tag{4.2}$$

Then we define the indirect utility function by

$$V(p,q,m) := U(x^d(p,q,m), y^d(p,q,m)) \quad \forall (p,q,m). \tag{4.3}$$

Consider a policy which changes q from $\bar{q}$ to $\bar{q}'$. By definition, the compensating variation is CV such that

$$V(\bar{p}, \bar{q}', \bar{m} - \text{CV}) = V(\bar{p}, \bar{q}, \bar{m}) \tag{4.4}$$

while the equivalent variation is EV such that

$$V(\bar{p}, \bar{q}, \bar{m} + \text{EV}) = V(\bar{p}, \bar{q}', \bar{m}). \tag{4.5}$$

We want to somehow compute CV and EV based on the observation of people's behavior in the market.

4.1 Expenditure function

To characterize CV and EV, we consider, again, the expenditure minimization problem. In the present context, the expenditure minimization problem is formulated as

$$\min_{x,y}\ px + y \quad \text{s.t. } (x, y) \in I(q, u) := \{(x, y) \in \mathbb{R}^2_+ \mid U(x, y, q) = u\} \tag{4.6}$$

for each $u \in \mathbb{R}$. Denote by $x^c(p, q, u), y^c(p, q, u)$ the compensated demand functions, namely,

$$(x^c(p, q, u), y^c(p, q, u)) \in \operatorname*{arg\,min}_{(x,y)\in I(q,u)} \{px + y\} \quad \forall (p, q, u). \tag{4.7}$$

Then we define the expenditure function by

$$e(p, q, u) := px^c(p, q, u) + y^c(p, q, u) \quad \forall (p, q, u). \tag{4.8}$$

The following results are straightforward generalizations of Theorem 3.1, Theorem 3.2, and Theorem 3.3, respectively.

> Theorem 4.1: **Demands, indirect utility, and expenditure**
>
> For any p, q, and m,
>
> (a) $x^c(p, q, V(p, q, m)) = x^d(p, q, m)$,
>
> (b) $e(p, q, V(p, q, m)) = m$.
>
> Moreover,
>
> (c) $x^d(p, q, e(p, q, u)) = x^c(p, q, u)$,
>
> (d) $V(p, q, e(p, q, u)) = u$
>
> for all p, q, and u.

Proof. Left as an exercise. ∎

> Theorem 4.2: **Shephard's lemma**
>
> For any p, q, and u,
>
> $$\frac{\partial e(p, q, u)}{\partial p} = x^c(p, q, u). \tag{4.9}$$

Proof. Left as an exercise. ∎

Combining Theorem 4.1 and Theorem 4.2 yields the following result.

Theorem 4.3: CV and EV for quantity-changing policies

Consider a public policy which changes q from $\bar{q}$ to $\bar{q}'$. The compensating variation associated with this policy may be computed by

$$\mathrm{CV} = e(\bar{p},\bar{q},\bar{u}) - e(\bar{p},\bar{q}',\bar{u}) = \int_{\bar{q}'}^{\bar{q}} \frac{\partial e(\bar{p},q,\bar{u})}{\partial q} dq, \tag{4.10}$$

where $\bar{u} := V(\bar{p},\bar{q},\bar{m})$. Similarly, we can compute the equivalent variation by

$$\mathrm{EV} = e(\bar{p},\bar{q},\bar{u}') - e(\bar{p},\bar{q}',\bar{u}') = \int_{\bar{q}'}^{\bar{q}} \frac{\partial e(\bar{p},q,\bar{u}')}{\partial q} dq, \tag{4.11}$$

where $\bar{u}' := V(\bar{p},\bar{q}',\bar{m})$.

Proof. Left as an exercise. ■

The term $\partial e/\partial q$ in (4.10) and (4.11) captures the marginal willingness to pay for quantity-changing policy, just as $\partial e/\partial p$ in (3.10) and (3.11) does for price-changing policy. To see this, let us say q is the quality of drinkable water supplied by a public utility in your city. Suppose you are not completely satisfied with the water quality and hence occasionally have to buy bottled water. If the quality q improves, then you would be able to achieve the same level of utility with less frequent purchase of bottled water. As a consequence, you will be able to save those money which you would otherwise have to spend for bottled water. By definition, this is the maximum amount of money you would be willing to pay in exchange for improving q.

Unlike $\partial e/\partial p$, we can not observe $\partial e/\partial q$ in general. Recall that $\partial e/\partial p$ is not directly observable either. But we know from Shephard's lemma that we can compute $\partial e/\partial p$ from demand function. Then the question is if there exists a theoretical analogue to Shephard's lemma for quantity-changing policies. The answer is (conditional) yes, as we show in the next theorem. Unfortunately, however, this result is not as useful as Shephard's lemma in itself.

Theorem 4.4

For any (p,q,u),

$$\frac{\partial e(p,q,u)}{\partial q} = -\frac{U_q(x^c(p,q,u),y^c(p,q,u),q)}{U_y(x^c(p,q,u),y^c(p,q,u),q)}, \tag{4.12}$$

where

$$U_q(x,y,q) := \frac{\partial U(x,y,q)}{\partial q}, \quad U_x(x,y,q) := \frac{\partial U(x,y,q)}{\partial x}. \tag{4.13}$$

Proof. By definition of expenditure function,

$$e(p,q,u) = px^c(p,q,u) + y^c(p,q,u) \quad \forall (p,q,u). \tag{4.14}$$

Differentiating both sides of the equation with respect to q, we have

$$\frac{\partial e(p,q,u)}{\partial q} = p\frac{\partial x^c(p,q,u)}{\partial q} + \frac{\partial y^c(p,q,u)}{\partial q}. \tag{4.15}$$

On the other hand, by definition of compensated demand functions,

$$U(x^c(p,q,u), y^c(p,q,u), q) = u \quad \forall (p,q,u). \tag{4.16}$$

Again, differentiate this equation with respect to q to obtain

$$\begin{aligned} U_x(x^c(p,q,u), y^c(p,q,u), q)\frac{\partial x^c(p,q,u)}{\partial q} \\ + U_y(x^c(p,q,u), y^c(p,q,u), q)\frac{\partial y^c(p,q,u)}{\partial q} \\ + U_q(x^c(p,q,u), y^c(p,q,u), q) = 0. \end{aligned} \tag{4.17}$$

It follows from the first-order condition of expenditure minimization (or utility maximization) that

$$\frac{U_x(x^c(p,q,u), y^c(p,q,u), q)}{U_y(x^c(p,q,u), y^c(p,q,u), q)} = p. \tag{4.18}$$

Combining (4.15), (4.17), and (4.18) yields the result. ■

Theorem 4.4 states that the marginal willingness to pay for quantity-changing policies is closely related to the marginal rate of substitution between public good q and the numéraire good y. While this result is of interest in its own right, it is not really useful because the right-hand side of (4.12) cannot be computed from observable data (unless some additional structure is imposed).

4.2 Weak complementarity

When the quantity or quality of public good is affected by a public policy, no general procedure exists for inferring its welfare impact from observable data. This does not mean, though, that it is impossible to construct an appropriate welfare measure for quantity-changing policies. We can still recover the welfare impact from people's choice in the market if their preference satisfies certain conditions. One such condition can be found in the notion of *weak complementarity*, which was coined by Mäler (1974).

Definition 4.1: Weak complements

Consider an individual whose preference is represented by a utility function $U(x,y,q)$. We say that x is a *weak complement* to q for this individual if

$$\left.\frac{\partial U(x,y,q)}{\partial q}\right|_{x=0} = 0 \quad \forall (y,q). \tag{4.19}$$

The definition of weak complementarity is easy to interpret. It basically says that people do not care about q when x is not consumed at all. This assumption is intuitively appealing in many cases of interest. Suppose, as an example, that q is the quality of natural park located in your city and x represents how frequently you visit the park. It seems reasonable to assume that the quality of the park is irrelevant if you never visit there. Similarly, suppose that q is the number of publicly available TV channels and x is the number of TV sets you buy per decade. How many TV channels offered by public broadcasting system does not matter when you do not own a TV set in the first place. If the relationship between x and q is of this sort, we say that they are weak complements.

Two remarks are in order. First, (4.19) does not hold when x is a *substitute* to q. Let us say, for instance, that x is the consumption of bottled water and q is the quality of drinkable water supplied by the government. Then x and q are likely to be substitutes in the sense that the demand for x would decrease when q improves, and vice versa. In this case, the quality of public water does matter and even more so when the bottled water is not consumed at all. We will later discuss how this case can be handled under the so-called household production model.

Second, the assumption of weak complementarity (or revealed preference methods in general) forces us to exclusively focus on *use value* of public goods. The definition of weak complementarity presumes that improving the quality of public goods is valuable only because we can make use of them for other purposes. In some cases, however, people may value a public good in and of itself. For example, people may care if a cultural heritage site is preserved or not even though they never visit the site. Or people might find it valuable to protect endangered species even though those species are of no direct use in daily lives. In these cases, x and q cannot be weak complements because the level of q affects the utility even when $x = 0$.

On top of weak complementarity, we need another condition called *non-essentiality*. We say that a good is non-essential if one can stop consuming the good when its price is sufficiently high. In other words, a good is said to be non-essential if there exists a price level above which people completely shift their consumption of the good to some substitute. The following definition formalizes the idea.

Definition 4.2: **Non-essential goods**

Consider an individual whose preference is represented by a utility function $U(x, y, q)$. We say that x is a *non-essential good* for this individual if for each (q, u), there exists $p^* \in \mathbb{R}_{++} \cup \{+\infty\}$ such that

$$x^c(p, q, u) = 0 \quad \forall p \geq p^*, \tag{4.20}$$

where x^c is the compensated demand for x. The price level p^* is often called the *choke price*.

Compared with weak complementarity, the assumption of non-essentiality is less restrictive. Most of the goods existing around us are in fact non-essential. If the price of TV sets becomes sufficiently high, for example, then consumer's demand for this good would be eventually choked off. This is because one could easily live without a TV set, killing time by browsing around the Internet instead. TV sets are therefore a non-essential good. When traveling to a natural park costs a lot, people will stop visiting the park and start spending their holidays in other places. Hence, visiting a park is another example of non-essential good, and the list goes on.

Before moving on to the main part of this section, we present a useful lemma, which we will use in the analysis that follows.

Lemma 4.1

Suppose that

(a) x is a weak complement to q, and

(b) x is a non-essential good.

Then

$$\lim_{p\to\infty} \frac{\partial e(p,q,u)}{\partial q} = 0 \quad \forall (q,u). \tag{4.21}$$

Proof. Since x is a non-essential good,

$$\lim_{p\to\infty} x^c(p,q,u) = 0 \quad \forall (q,u). \tag{4.22}$$

This in turn implies

$$\lim_{p\to\infty} \frac{\partial U(x^c(p,q,u), y^c(p,q,u), q)}{\partial q} = 0 \quad \forall (q,u) \tag{4.23}$$

because x is a weak complement to q. It then follows from Theorem 4.4 that

$$\lim_{p\to\infty} \frac{\partial e(p,q,u)}{\partial q} = -\lim_{p\to\infty} \frac{U_q(x^c(p,q,u), y^c(p,q,u), q)}{U_y(x^c(p,q,u), y^c(p,q,u), q)} = 0 \quad \forall (q,u), \tag{4.24}$$

as claimed. ■

The interpretation of this lemma should be straightforward. If x is a non-essential good, the compensated demand for this good is completely choked off for a sufficiently high price. If furthermore x is a weak complement to q, then the level of q does not matter when x is not consumed. As a consequence, the (marginal) willingness to pay for quantity-changing policies should converge to zero as $p \to \infty$. Technically, this is equivalent to saying that the expenditure function $e(p,q,u)$ should be constant as a function of q if p is sufficiently high.

We are now ready to state the main result of this section.

Theorem 4.5: **Compensated-demand representation of CV and EV**

Consider a public policy which changes q from $\bar{q}$ to $\bar{q}'$. If

(a) x is a weak complement to q, and

(b) x is a non-essential good,

then we may rewrite CV and EV in Theorem 4.3 as

$$\text{CV} = \int_{\bar{p}}^{\infty} x^c(p, \bar{q}', \bar{u})dp - \int_{\bar{p}}^{\infty} x^c(p, \bar{q}, \bar{u})dp \tag{4.25}$$

and

$$\text{EV} = \int_{\bar{p}}^{\infty} x^c(p, \bar{q}', \bar{u}')dp - \int_{\bar{p}}^{\infty} x^c(p, \bar{q}, \bar{u}')dp, \tag{4.26}$$

respectively.

Proof. By definition of CV,

$$\begin{aligned}
\text{CV} &= e(\bar{p}, \bar{q}, \bar{u}) - e(\bar{p}, \bar{q}', \bar{u}) \\
&= e(\bar{p}, \bar{q}, \bar{u}) - e(\tilde{p}, \bar{q}, \bar{u}) \\
&\quad + e(\tilde{p}, \bar{q}, \bar{u}) - e(\tilde{p}, \bar{q}', \bar{u}) \\
&\quad + e(\tilde{p}, \bar{q}', \bar{u}) - e(\bar{p}, \bar{q}', \bar{u}) \\
&= \int_{\bar{p}}^{\tilde{p}} x^c(p, \bar{q}', \bar{u})dp - \int_{\bar{p}}^{\tilde{p}} x^c(p, \bar{q}, \bar{u})dp + \int_{\bar{q}'}^{\bar{q}} \frac{\partial e(\tilde{p}, q, \bar{u})}{\partial q} dq
\end{aligned} \tag{4.27}$$

for any $\tilde{p}$. We know from Lemma 4.1 that

$$\lim_{\tilde{p}\to\infty} \int_{\bar{q}'}^{\bar{q}} \frac{\partial e(\tilde{p}, q, \bar{u})}{\partial q} dq = \int_{\bar{q}'}^{\bar{q}} \left(\lim_{\tilde{p}\to\infty} \frac{\partial e(\tilde{p}, q, \bar{u})}{\partial q} \right) dq = 0. \tag{4.28}$$

Hence, taking the limit of (4.27) as $\tilde{p} \to \infty$, we have

$$\text{CV} = \int_{\bar{p}}^{\infty} x^c(p, \bar{q}', \bar{u})dp - \int_{\bar{p}}^{\infty} x^c(p, \bar{q}, \bar{u})dp, \tag{4.29}$$

which is (4.25). Use exactly the same argument for EV and obtain (4.26). ■

To interpret the statement of Theorem 4.5, observe first that when x is a weak complement to q, the value of q is completely internalized in the value of x. For instance, let us say that q is the number of TV channels and x is the number of TV sets you own. When q increases, your TV sets would become more valuable since you would then be able to watch a wider variety of TV shows. When q decreases, on the other hand, you would benefit less from owning the same number of TV sets. Suppose, for another example, that q is the quality of a natural park and x represents how frequently you visit the park. When q improves, then you would obtain a higher level of welfare from visiting the park even if the frequency of your visit remains as before. When q diminishes, on the other hand, you would be able to gain a smaller benefit from the same number of visits to the park.

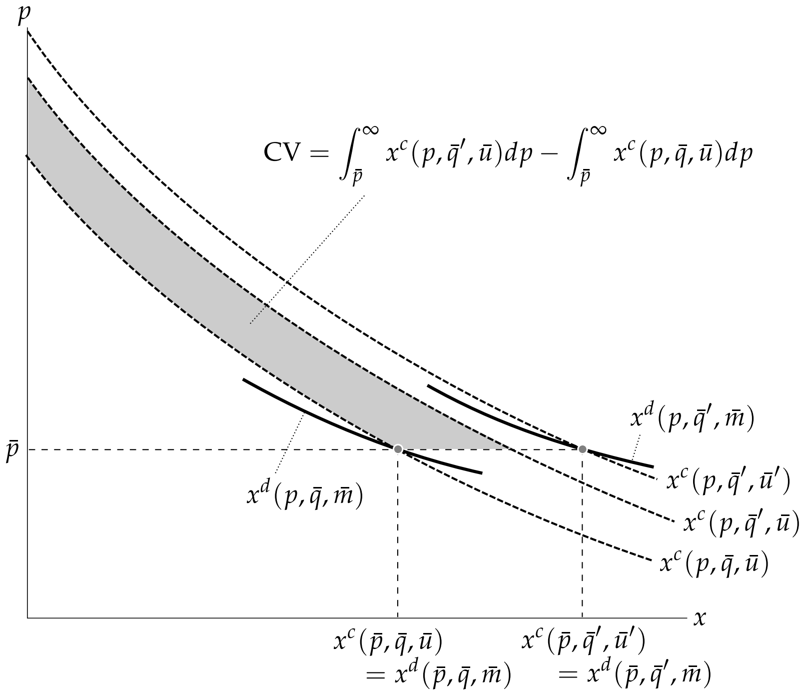

Figure 2: CV under weak complementarity

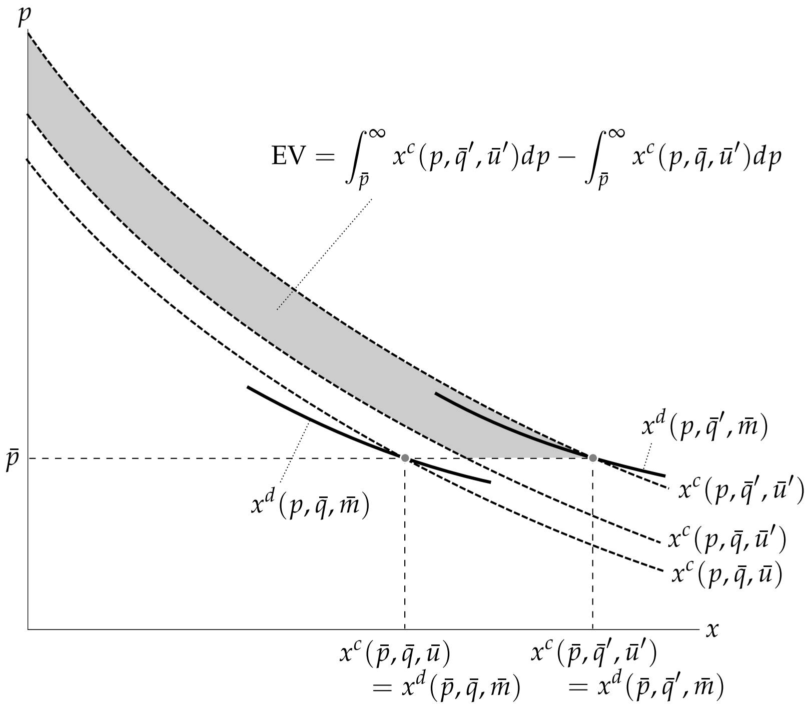

Figure 3: EV under weak complementarity

This observation suggests that we should be able to infer the welfare impact caused by a change in q if we compare the value of x before and after q changes. Theorem 4.5 formalizes this idea. In (4.25), the term

$$\int_{\bar{p}}^{\infty} x^c(p,\bar{q},\bar{u})dp = \lim_{p\to\infty} e(p,\bar{q},\bar{u}) - e(\bar{p},\bar{q},\bar{u}) \tag{4.30}$$

measures in monetary terms the level of welfare people gain from consuming x when $q = \bar{q}$. On the other hand, the term

$$\int_{\bar{p}}^{\infty} x^c(p,\bar{q}',\bar{u})dp = \lim_{p\to\infty} e(p,\bar{q}',\bar{u}) - e(\bar{p},\bar{q}',\bar{u}) \tag{4.31}$$

is the level of welfare people gain from consuming the same amount of good x, but under a different level of public good (i.e., when $q = \bar{q}'$). The difference between these two terms should capture the welfare impact caused by a change in q from $\bar{q}$ to $\bar{q}'$.

Theorem 4.5 shows that we can compute CV and EV if we have the knowledge of compensated demand function. But how can we move forward from here? One might think that we can use Theorem 3.5, which suggests that the compensated demand function can be approximated by the ordinary demand function and the approximation error is determined based on the income elasticity of demand and the consumer's surplus. Unfortunately, Theorem 3.5 is not directly applicable. To see this, define the associated consumer's surplus by

$$\Delta\text{CS} := \int_{\bar{p}}^{\infty} x^d(p,\bar{q}',\bar{m})dp - \int_{\bar{p}}^{\infty} x^d(p,\bar{q},\bar{m})dp \tag{4.32}$$

Notice that

$$\int_{\bar{p}}^{\infty} x^d(p,\bar{q},\bar{m})dp \approx \int_{\bar{p}}^{\infty} x^c(p,\bar{q},\bar{u})dp \tag{4.33}$$

and

$$\int_{\bar{p}}^{\infty} x^d(p,\bar{q}',\bar{m})dp \approx \int_{\bar{p}}^{\infty} x^c(p,\bar{q}',\bar{u}')dp, \tag{4.34}$$

where the approximation comes from Theorem 3.5. This implies

$$\begin{aligned}
\text{CV} &= \int_{\bar{p}}^{\infty} x^c(p,\bar{q}',\bar{u})dp - \int_{\bar{p}}^{\infty} x^c(p,\bar{q},\bar{u})dp \\
&= \int_{\bar{p}}^{\infty} x^c(p,\bar{q}',\bar{u}')dp - \int_{\bar{p}}^{\infty} x^c(p,\bar{q},\bar{u})dp \\
&\quad + \int_{\bar{p}}^{\infty} x^c(p,\bar{q}',\bar{u})dp - \int_{\bar{p}}^{\infty} x^c(p,\bar{q}',\bar{u}')dp \\
&\approx \int_{\bar{p}}^{\infty} x^d(p,\bar{q}',\bar{m})dp - \int_{\bar{p}}^{\infty} x^d(p,\bar{q},\bar{m})dp \\
&\quad + \int_{\bar{p}}^{\infty} x^c(p,\bar{q}',\bar{u})dp - \int_{\bar{p}}^{\infty} x^c(p,\bar{q}',\bar{u}')dp \\
&= \Delta\text{CS} - \underbrace{\int_{\bar{p}}^{\infty} (x^c(p,\bar{q}',\bar{u}') - x^c(p,\bar{q}',\bar{u}))dp}_{\text{extra error term}}.
\end{aligned} \tag{4.35}$$

Since $x^c(p,\bar{q}',\bar{u}')$ does not coincide with $x^c(p,\bar{q}',\bar{u})$ in general, CV cannot be well approximated by ΔCS even if the approximations in (4.33) and (4.34) are fairly good. This fact can be easily seen in Figure 2. Similarly, as is clear in Figure 3, we have

$$\text{EV} \approx \Delta\text{CS} - \underbrace{\int_{\bar{p}}^{\infty} (x^c(p,\bar{q},\bar{u}') - x^c(p,\bar{q},\bar{u}))dp}_{\text{extra error term}}, \tag{4.36}$$

where $x^c(p,\bar{q},\bar{u}') \neq x^c(p,\bar{q},\bar{u})$ in general.

If there is no income effect (i.e., if $\partial x^d/\partial m = 0$), the compensated demand function coincides with the demand function because

$$\begin{aligned} \frac{\partial x^c(p,q,u)}{\partial p} &= \left.\frac{\partial x^d(p,q,m)}{\partial p}\right|_{m=e(p,q,u)} + \left.\frac{\partial x^d(p,q,m)}{\partial m}\right|_{m=e(p,q,u)} \frac{\partial e(p,q,u)}{\partial p} \\ &= \left.\frac{\partial x^d(p,q,m)}{\partial p}\right|_{m=e(p,q,u)}, \end{aligned} \tag{4.37}$$

where first line uses Theorem 4.1. Moreover, $x^c(p,q,u)$ becomes independent of u because

$$\frac{\partial x^c(p,q,u)}{\partial u} = \left.\frac{\partial x^d(p,q,m)}{\partial m}\right|_{m=e(p,q,u)} \frac{\partial e(p,q,u)}{\partial u} = 0, \tag{4.38}$$

where the first equality follows from Theorem 4.1. This suggests that $x^c(p,\bar{q}',\bar{u}') = x^c(p,\bar{q}',\bar{u})$ and $x^c(p,\bar{q},\bar{u}') = x^c(p,\bar{q},\bar{u})$. Then CV and EV can be computed as

$$\text{CV} = \text{EV} = \int_{\bar{p}}^{\infty} x^d(p,\bar{q}',\bar{m})dp - \int_{\bar{p}}^{\infty} x^d(p,\bar{q},\bar{m})dp = \Delta\text{CS}. \tag{4.39}$$

Therefore, it is reasonable to use consumer's surplus as an approximate welfare measure for quantity-changing policies when the income effect is sufficiently small.

In practice, researchers frequently focus on those policies which entail an all-or-nothing choice about the supply of public goods. Consider, for example, a public policy which ruins a natural park for construction of new highways. In other words, this policy changes q from $\bar{q} > 0$ to $\bar{q}' := 0$. Assume that x, the number of visitors to the park, would only be positive when $q > 0$ so that

$$x^d(p,\bar{q}',\bar{m}) = x^c(p,\bar{q}',\bar{u}) = 0 \quad \forall p. \tag{4.40}$$

Then the extra error term in (4.35) vanishes and

$$\text{CV} = -\int_{\bar{p}}^{\infty} x^c(p,\bar{q},\bar{u})dp \approx -\int_{\bar{p}}^{\infty} x^d(p,\bar{q},\bar{m})dp = \Delta\text{CS}, \tag{4.41}$$

where the approximation is only based on Theorem 3.5. Hence, in this case, one can reasonably use consumer's surplus as an approximate welfare measure as long as the approximation error (3.27) is fairly small.

4.3 Household production model

Apart from weak complementarity, there is another important case in which we can infer the welfare impact of quantity-changing policies. The case we will consider in this section is most appropriately characterized by the *household production model*. To understand how this model helps, recall Theorem 4.4, which states that we could compute the willingness to pay for a change in q if we happen to know the marginal rate of substitution between q and x. This is not possible in general because it requires the knowledge of preference. In some cases, however, the relationship between q and x is solely determined by technology, rather than by preference.

In order to fix the context, consider a developing country where publicly supplied water is of poor quality. People living in this country use water filters so that they can make the water drinkable. In other words, drinkable water is 'produced' within each household by combining the water publicly supplied by the government with the water filters privately purchased in the market. This household production process may be formally described by

$$w = f(x, q), \tag{4.42}$$

where w is the amount of drinkable water, x is the annual consumption of water filters, and q is the average water quality in the region. The function f — a *household production function* — represents a technical relationship existing between x and q. When the quality of public water is poor, water filters need to be replaced frequently. But if the government improves the water quality, people would need to replace water filters less frequently to produce the same amount of drinkable water. In other words, x and q are substitutes and the substitutability is captured by the household production function f. It is worth mentioning that f is likely to be observable because the relationship between x and q is purely technical, independent of people's preference. This greatly simplifies our task.

To illustrate the point, let us reinterpret the utility maximization problem (4.1) in the context of household production model. In our example, neither water filters nor public water is directly consumed by households. What really matters for consumer's utility is the amount of drinkable water. The utility function is hence more appropriately specified as

$$\tilde{U}(w, y). \tag{4.43}$$

The associated utility maximization problem is

$$\max_{w,y} \tilde{U}(w, y) \ \text{ s.t. } (w, y) \in B(p, m, q), \tag{4.44}$$

where

$$B(p, m, q) := \{(w, y) \in \mathbb{R}^2_+ \mid px + y = m \text{ and } w = f(x, q)\}. \tag{4.45}$$

This problem might look different from the one we had in mind in the preceding section. But if we define

$$U(x, y, q) := \tilde{U}(f(x, q), y) \quad \forall (x, y, q), \tag{4.46}$$

we may rewrite (4.44) as

$$\max_{x,y} U(x,y,q) \;\text{ s.t. } (x,y) \in B(p,m) := \{(x,y) \in \mathbb{R}^2_+ \mid px + y = m\}, \tag{4.47}$$

which is nothing but (4.1). Therefore, the analysis in the preceding section all carries over here.

In particular, by Theorem 4.4,

$$\begin{aligned}\frac{\partial e(p,q,u)}{\partial q} &= -p\frac{U_q(x^c(p,q,u),y^c(p,q,u),q)}{U_x(x^c(p,q,u),y^c(p,q,u),q)} \\ &= -p\frac{\tilde{U}_w(f(x^c(p,q,u),q),y^c(p,q,u))f_q(x^c(p,q,u),q)}{\tilde{U}_w(f(x^c(p,q,u),q),y^c(p,q,u))f_x(x^c(p,q,u),q)} \\ &= -p\frac{f_q(x^c(p,q,u),q)}{f_x(x^c(p,q,u),q)}\end{aligned} \tag{4.48}$$

for any (p,q,u). It then follows from Theorem 4.3 that

$$\text{CV} = \int_{\bar{q}'}^{\bar{q}} \frac{\partial e(\bar{p},q,\bar{u})}{\partial q}dq = p\int_{\bar{q}}^{\bar{q}'} \frac{f_q(x^c(\bar{p},q,\bar{u}),q)}{f_x(x^c(\bar{p},q,\bar{u}),q)}dq \tag{4.49}$$

and

$$\text{EV} = \int_{\bar{q}'}^{\bar{q}} \frac{\partial e(\bar{p},q,\bar{u}')}{\partial q}dq = p\int_{\bar{q}}^{\bar{q}'} \frac{f_q(x^c(\bar{p},q,\bar{u}'),q)}{f_x(x^c(\bar{p},q,\bar{u}'),q)}dq. \tag{4.50}$$

Observe that both CV and EV are characterized by f_q/f_x, the marginal rate of *technical* substitution. Rewriting CV and EV this way helps a lot because the knowledge of preference is not required in these expressions. All we need to do is to estimate the technical relationship between x and q — how often people would have to change water filters for a given quality of public water. In simple cases, we do not even need to have the knowledge of demand function. For instance, suppose that the household production function f is estimated as

$$f(x,q) = x + g(q) \tag{4.51}$$

for some function g. Then the marginal rate of technical substitution is

$$\frac{f_q(x,q)}{f_x(x,q)} = g'(q) \tag{4.52}$$

and therefore, in this case, CV and EV are simply given by

$$\text{CV} = \text{EV} = p\int_{\bar{q}}^{\bar{q}'} g'(q)dq = p\left(g(\bar{q}') - g(\bar{q})\right), \tag{4.53}$$

which can be computed based on the knowledge of g.

The household production framework is also useful for determining the upper and lower bounds of CV and EV. Assuming that the household production function $w = f(x,q)$ is strictly increasing in x, let $x = f^{-1}(w,q)$ be the inverse function in terms of x. Geometrically, the graph of $f^{-1}(w,\cdot)$ represents an isoquant curve, the set of all combinations of x and q with which one can produce w. With the knowledge of this isoquant curve, the upper and lower bounds of CV and EV can be computed as follows.

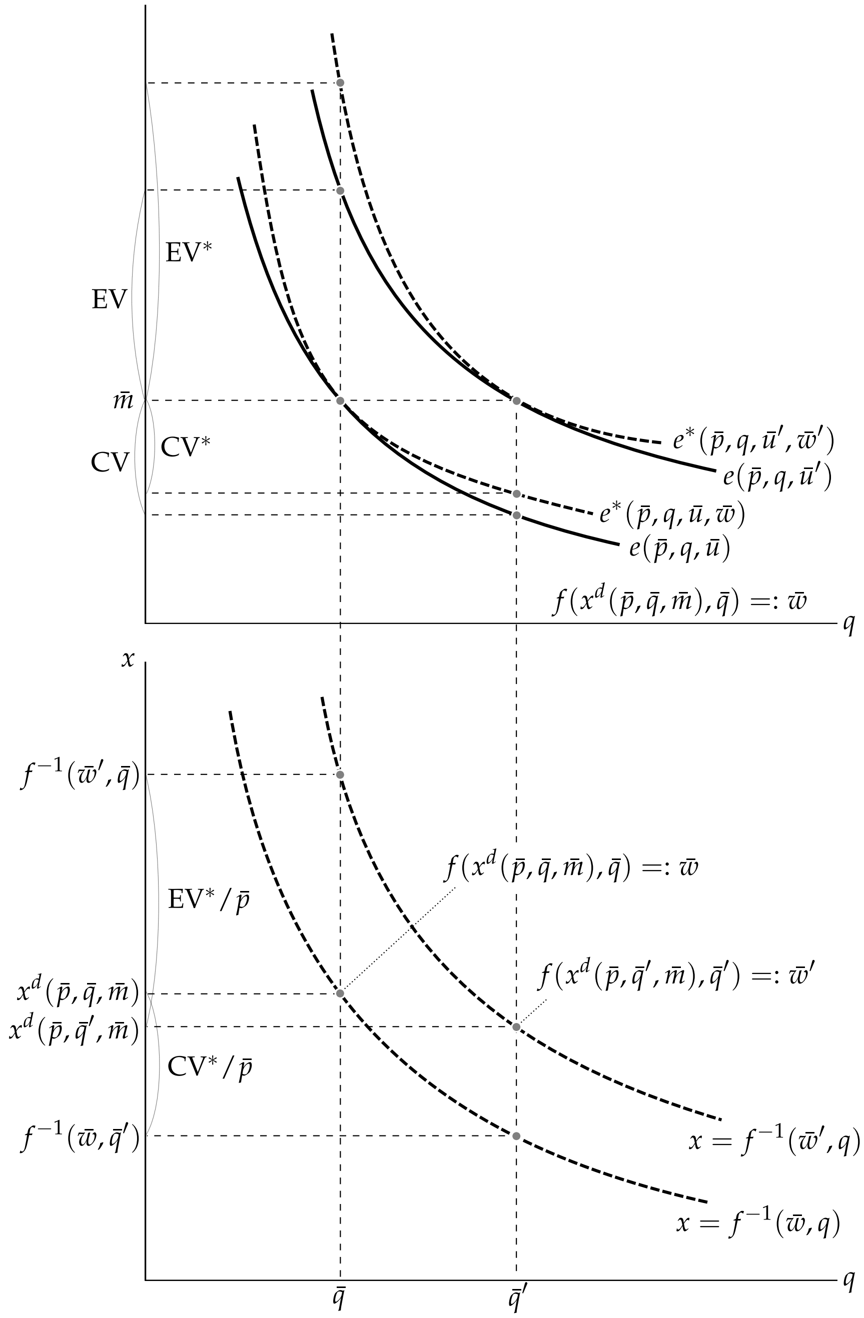

Figure 4: Lower and upper bounds of CV and EV

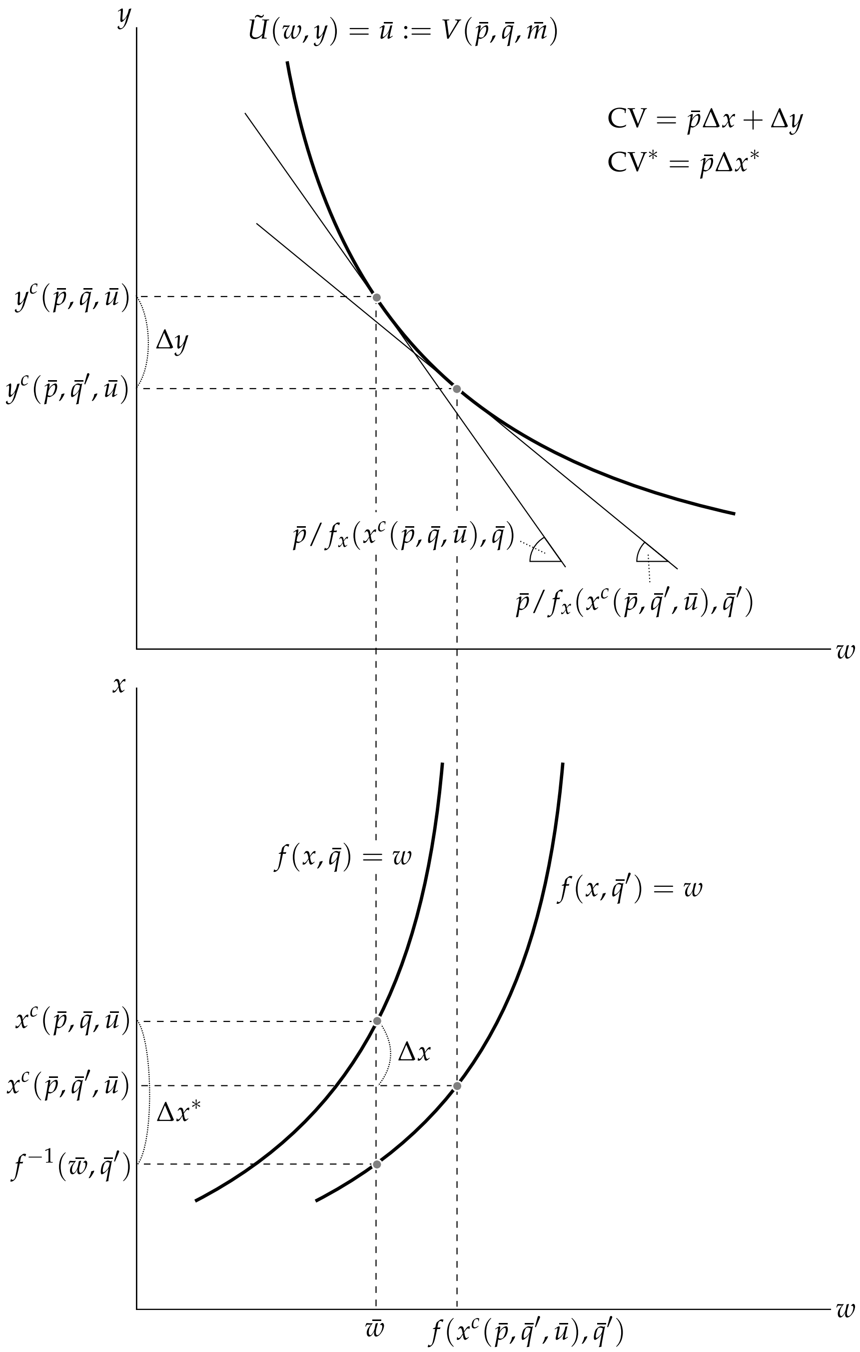

Figure 5: CV vs CV*

Theorem 4.6: **Lower and Upper bounds of CV and EV**

Consider a household production model defined by (4.44). For a public policy which changes q from $\bar{q}$ to $\bar{q}'$, the associated CV is bounded below by

$$\mathrm{CV}^* := \bar{p}\left(x^d(\bar{p},\bar{q},\bar{m}) - f^{-1}(\bar{w},\bar{q}')\right) \leq \mathrm{CV}, \tag{4.54}$$

where $\bar{w} := f(x^d(\bar{p},\bar{q},\bar{m}),\bar{q})$. The associated EV is bounded above by

$$\mathrm{EV}^* := \bar{p}\left(f^{-1}(\bar{w}',\bar{q}) - x^d(\bar{p},\bar{q}',\bar{m})\right) \geq \mathrm{EV}, \tag{4.55}$$

where $\bar{w}' := f(x^d(\bar{p},\bar{q}',\bar{m}),\bar{q}')$.

Proof. Define

$$e^*(p,q,u,w) := \min_{x,y}\{px + y \mid \tilde{U}(f(x,q),y) = u \text{ and } f(x,q) = w\} \tag{4.56}$$

for each (p,q,u,w) and put

$$\mathrm{CV}^* := e^*(\bar{p},\bar{q},\bar{u},\bar{w}) - e^*(\bar{p},\bar{q}',\bar{u},\bar{w}), \tag{4.57}$$

$$\mathrm{EV}^* := e^*(\bar{p},\bar{q},\bar{u}',\bar{w}') - e^*(\bar{p},\bar{q}',\bar{u}',\bar{w}'). \tag{4.58}$$

It should be easy to see that

$$\begin{aligned} e(p,q,u) &:= \min_{x,y}\{px + y \mid \tilde{U}(f(x,q),y) = u\} \\ &\leq \min_{x,y}\{px + y \mid \tilde{U}(f(x,q),y) = u \text{ and } f(x,q) = w\} \\ &= e^*(p,q,u,w) \quad \forall (p,q,u,w), \end{aligned} \tag{4.59}$$

where the inequality is replaced by equality when $w = f(x^c(p,q,u),q)$. In particular,

$$e(\bar{p},\bar{q}',\bar{u}) \leq e^*(\bar{p},\bar{q}',\bar{u},\bar{w}) \text{ and } e(\bar{p},\bar{q},\bar{u}) = e^*(\bar{p},\bar{q},\bar{u},\bar{w}), \tag{4.60}$$

where

$$\bar{w} := f(x^d(\bar{p},\bar{q},\bar{m}),\bar{q}) = f(x^c(\bar{p},\bar{q},\bar{u}),\bar{q}). \tag{4.61}$$

Hence,

$$\begin{aligned} \mathrm{CV}^* &= e^*(\bar{p},\bar{q},\bar{u},\bar{w}) - e^*(\bar{p},\bar{q}',\bar{u},\bar{w}) \\ &\leq e(\bar{p},\bar{q},\bar{u}) - e(\bar{p},\bar{q}',\bar{u}) = \mathrm{CV}. \end{aligned} \tag{4.62}$$

Similarly,

$$e(\bar{p},\bar{q},\bar{u}') \leq e^*(\bar{p},\bar{q},\bar{u}',\bar{w}') \text{ and } e(\bar{p},\bar{q}',\bar{u}') = e^*(\bar{p},\bar{q}',\bar{u}',\bar{w}'), \tag{4.63}$$

where

$$\bar{w}' := f(x^d(\bar{p},\bar{q}',\bar{m}),\bar{q}) = f(x^c(\bar{p},\bar{q}',\bar{u}'),\bar{q}'), \tag{4.64}$$

which implies

$$\begin{aligned}\mathrm{EV}^* &= e^*(\bar{p},\bar{q},\bar{u}',\bar{w}') - e^*(\bar{p},\bar{q}',\bar{u}',\bar{w}') \\ &\geq e(\bar{p},\bar{q},\bar{u}') - e(\bar{p},\bar{q}',\bar{u}') = \mathrm{EV}.\end{aligned} \tag{4.65}$$

On the other hand, since $\bar{w} = f(x^d(\bar{p},\bar{q},\bar{m}),\bar{q}) = f(x^c(\bar{p},\bar{q},\bar{u}),\bar{q})$, we have

$$\tilde{U}(\bar{w}, y^c(\bar{p},\bar{q},\bar{u})) = \tilde{U}(f(x^c(\bar{p},\bar{q},\bar{u}),\bar{q}), y^c(\bar{p},\bar{q},\bar{u})) = \bar{u}, \tag{4.66}$$

which suggests that

$$\begin{aligned}e^*(\bar{p},\bar{q}',\bar{u},\bar{w}) &= \min_{x,y}\{\bar{p}x + y \mid \tilde{U}(f(x,\bar{q}'),y) = \bar{u} \text{ and } f(x,\bar{q}') = \bar{w}\} \\ &= \min_{y}\{\bar{p}f^{-1}(\bar{w},\bar{q}') + y \mid \tilde{U}(\bar{w},y) = \bar{u}\} \\ &= \bar{p}f^{-1}(\bar{w},\bar{q}') + y^c(\bar{p},\bar{q},\bar{u}).\end{aligned} \tag{4.67}$$

Hence,

$$\begin{aligned}\mathrm{CV}^* &= e^*(\bar{p},\bar{q},\bar{u},\bar{w}) - e^*(\bar{p},\bar{q}',\bar{u},\bar{w}) \\ &= e(\bar{p},\bar{q},\bar{u}) - e^*(\bar{p},\bar{q}',\bar{u},\bar{w}) \\ &= (\bar{p}x^c(\bar{p},\bar{q},\bar{u}) + y^c(\bar{p},\bar{q},\bar{u})) - (\bar{p}f^{-1}(\bar{w},\bar{q}') + y^c(\bar{p},\bar{q},\bar{u})) \\ &= \bar{p}\left(x^d(\bar{p},\bar{q},\bar{m}) - f^{-1}(\bar{w},\bar{q}')\right).\end{aligned} \tag{4.68}$$

Similarly, since $\bar{w}' = f(x^d(\bar{p},\bar{q}',\bar{m}),\bar{q}') = f(x^c(\bar{p},\bar{q}',\bar{u}'),\bar{q}')$,

$$\tilde{U}(\bar{w}', y^c(\bar{p},\bar{q}',\bar{u}')) = \tilde{U}(f(x^c(\bar{p},\bar{q}',\bar{u}'),\bar{q}), y^c(\bar{p},\bar{q}',\bar{u})') = \bar{u}', \tag{4.69}$$

which suggests that

$$\begin{aligned}e^*(\bar{p},\bar{q},\bar{u}',\bar{w}') &= \min_{x,y}\{\bar{p}x + y \mid \tilde{U}(f(x,\bar{q}),y) = \bar{u}' \text{ and } f(x,\bar{q}) = \bar{w}'\} \\ &= \min_{y}\{\bar{p}f^{-1}(\bar{w}',\bar{q}) + y \mid \tilde{U}(\bar{w}',y) = \bar{u}'\} \\ &= \bar{p}f^{-1}(\bar{w}',\bar{q}) + y^c(\bar{p},\bar{q}',\bar{u}').\end{aligned} \tag{4.70}$$

Hence,

$$\begin{aligned}\mathrm{EV}^* &= e^*(\bar{p},\bar{q},\bar{u}',\bar{w}') - e^*(\bar{p},\bar{q}',\bar{u}',\bar{w}') \\ &= e^*(\bar{p},\bar{q},\bar{u}',\bar{w}') - e(\bar{p},\bar{q}',\bar{u}') \\ &= (\bar{p}f^{-1}(\bar{w}',\bar{q}) + y^c(\bar{p},\bar{q}',\bar{u}')) - (\bar{p}x^c(\bar{p},\bar{q}',\bar{u}') + y^c(\bar{p},\bar{q}',\bar{u}')) \\ &= \bar{p}\left(f^{-1}(\bar{w}',\bar{q}) - x^d(\bar{p},\bar{q}',\bar{m})\right).\end{aligned} \tag{4.71}$$

Combining (4.62), (4.68), (4.65), and (4.71) yields the result. ■

This result, which was presented in a seminal paper by Bartik (1988), is based on the idea of *defensive expenditure*. Roughly speaking, defensive expenditure is the amount of money you need to spend for averting adverse effects of public bads. In our water-filter example, poor water quality (which is considered to be a kind of public bad) makes it necessary to buy water filters. Buying $x^d(\bar{p}, \bar{q}, \bar{m})$ units of water filters, you can produce $\bar{w} = f(x^d(\bar{p}, \bar{q}, \bar{m}), \bar{q})$ units of drinkable water. The defensive expenditure associated with this averting behavior is $\bar{p}x^d(\bar{p}, \bar{q}, \bar{m})$. When the water quality improves from $\bar{q}$ to $\bar{q}' > \bar{q}$, the same amount of drinkable water can be produced with less frequent purchase of water filters. More precisely, only $f^{-1}(\bar{w}, \bar{q}')$ units of water filters will be needed to produce $\bar{w}$ units of drinkable water. As a result, the associated defensive expenditure will decline to $\bar{p}f^{-1}(\bar{w}, \bar{q}')$. In other words, when q changes from $\bar{q}$ to $\bar{q}'$, you would be able to reduce defensive expenditure to the amount of $\mathrm{CV}^* := \bar{p}(x^d(\bar{p}, \bar{q}, \bar{m}) - f^{-1}(\bar{w}, \bar{q}'))$ while producing the same amount of drinkable water as before. This suggests that you should be willing to pay *at least* CV^* for a public policy which changes q from $\bar{q}$ to $\bar{q}'$. Therefore, CV^* is a lower bound of CV.

One might wonder why CV^* does not provide the exact welfare measure. The reason for CV^* being smaller than CV in general is that once q changes from $\bar{q}$ to $\bar{q}'$, you would not necessarily want to produce the same amount of w as before in terms of compensated demands. When the quality of public water improves, then drinkable water becomes effectively cheaper because a smaller amount of water filters would be needed per unit production of drinkable water. This means that in response to an increase in q, the relative price will be tilted in favor of lager (household) production of drinkable water. Hence, under the improved water quality $\bar{q}'$, reducing the consumption of water filters from $x^d(\bar{p}, \bar{q}, \bar{m})$ to $f^{-1}(\bar{w}, \bar{q}')$ is 'too much,' failing to minimize the total expenditure. By choosing x somewhere in between $x^d(\bar{p}, \bar{q}, \bar{m})$ and $f^{-1}(\bar{w}, \bar{q}')$ and by at the same time reducing consumption of y, you could achieve savings larger than CV^* without sacrificing the original level of utility. This is why the maximum amount of money you are willing to pay for q-improving policy, which is the definition of CV, should be larger than CV^*. See Figure 5 for a diagramatic illustration of this fact.

4.4 Empirical application

Consider an economy consisting of $N \in \mathbb{N}$ individuals. Assume that individual's demand function takes the form

$$x^d(p, q, m) = e^{\beta_p p + \beta_q q + \eta \ln(m) + \varepsilon}, \tag{4.72}$$

where ε captures an individual-specific preference parameter (or some idiosyncratic preference shock). Here, η is the income elasticity of demand. We further assume that the distribution of ε among individuals is well approximated by the Poisson distribution with $\mathbb{E}[e^{\varepsilon}] = e^{\beta_0}$. This means that when a sample is randomly drawn from the population, the probability of a particular x being observed is

$$\Pr(x^d(p, q, m) = x) = \frac{e^{-\lambda(p,q,m)}(\lambda(p, q, m))^x}{x!}, \tag{4.73}$$

where

$$\lambda(p, q, m) := \mathbb{E}[x^d(p, q, m)] = e^{\beta_0 + \beta_p p + \beta_q q + \eta \ln(m)}. \tag{4.74}$$

For simplicity, let us focus on a public policy which changes q from $\bar{q}$ to $\bar{q}' = 0$. Also, suppose that there are $K \in \mathbb{N}$ distinct combinations of $(\bar{p}, \bar{m})$ in the population and denote by $N_k \in \mathbb{N}$ the number of individuals whose price-income pair is $(\bar{p}_k, \bar{m}_k)$. Obviously, $\sum_{k=1}^{K} N_k = N$. Then, assuming the income elasticity η is sufficiently small, it follows from (4.41) that the associated compensating variation is given by

$$
\begin{aligned}
|\mathrm{CV}| &= \sum_{i=1}^{N} \int_{\bar{p}_i}^{\infty} x_i^c(p, \bar{q}, \bar{u}_i) dp \text{ where } \bar{u}_i := V(\bar{p}_i, \bar{q}, \bar{m}_i) \\
&\approx \sum_{i=1}^{N} \int_{\bar{p}_i}^{\infty} x_i^d(p, \bar{q}, \bar{m}_i) dp \\
&= N \sum_{k=1}^{K} \frac{N_k}{N} \left(\frac{1}{N_k} \sum_{i=1}^{N_k} \int_{\bar{p}_k}^{\infty} x_i^d(p, \bar{q}, \bar{m}_k) dp \right) \\
&= N \sum_{k=1}^{K} \frac{N_k}{N} \mathbb{E} \left[\int_{\bar{p}_k}^{\infty} x^d(p, \bar{q}, \bar{m}_k) dp \right], \qquad (4.75)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{E} \left[\int_{\bar{p}_k}^{\infty} x^d(p, \bar{q}, \bar{m}_k) dp \right] &= \int_{\bar{p}_k}^{\infty} \mathbb{E} \left[x^d(p, \bar{q}, \bar{m}_k) \right] dp \\
&= \int_{\bar{p}_k}^{\infty} e^{\beta_0 + \beta_p p + \beta_q \bar{q} + \eta \ln(\bar{m}_k)} dp \\
&= \lim_{p \to \infty} \frac{1}{\beta_p} e^{\beta_0 + \beta_p p + \beta_q \bar{q} + \eta \ln(\bar{m}_k)} - \frac{1}{\beta_p} e^{\beta_0 + \beta_p \bar{p}_k + \beta_q \bar{q} + \eta \ln(\bar{m}_k)} \\
&= -\lambda(\bar{p}_k, \bar{q}, \bar{m}_k) / \beta_p. \qquad (4.76)
\end{aligned}
$$

Hence,

$$
|\mathrm{CV}| \approx -N \frac{\lambda}{\beta_p}, \quad \text{where} \quad \lambda := \sum_{k=1}^{K} \frac{N_k}{N} \lambda(\bar{p}_k, \bar{q}, \bar{m}_k). \qquad (4.77)
$$

If we have reliable estimates for λ and β_p, which we denote by $\hat{\lambda}$ and $\hat{\beta}_p$ respectively, we obtain the estimated value of CV as

$$
|\widehat{\mathrm{CV}}| = -N \frac{\hat{\lambda}}{\hat{\beta}_p}. \qquad (4.78)
$$

To get a sense of how the estimation procedure works, suppose that we have an observation $(x_i, \bar{p}_i, \bar{m}_i)_{i=1}^{T}$ where $T \in \mathbb{N}$ is the sample size. As long as the sample size is sufficiently large, a good estimator for λ is the sample mean

$$
\hat{\lambda} := \frac{1}{T} \sum_{i=1}^{T} x_i. \qquad (4.79)
$$

Also, given the Poisson specification of the model, it is straightforward to obtain an estimate for β_p. Combining (4.73) with (4.74) yields the likelihood function

$$L(\beta_0, \beta_p, \eta | (x_i, \bar{p}_i, \bar{m}_i)_{i=1}^T) := \prod_{i=1}^{T} \frac{e^{-e^{\beta_0 + \beta_p \bar{p}_i + \beta_q \bar{q} + \eta \ln(\bar{m}_i)}} \left(e^{\beta_0 + \beta_p \bar{p}_i + \beta_q \bar{q} + \eta \ln(\bar{m}_i)}\right)^{x_i}}{x_i!}. \tag{4.80}$$

Notice that in this example, we cannot identify β_q since there is no variation in the level of q. The maximum-likelihood estimators are then computed as

$$(\hat{\beta}_0, \hat{\beta}_p, \hat{\eta}) \in \underset{\beta_0, \beta_p, \eta}{\arg\max}\, L(\beta_0, \beta_p, \eta | (x_i, \bar{p}_i, \bar{m}_i)_{i=1}^T). \tag{4.81}$$

For example, Haab and McConnel (2002) used a data set of McConnell (1986) and estimated the demand function for trips to New Bedford Beach in Massachusetts. The average annual trips to the beach was $\hat{\lambda} = 3.83$ and the price coefficient (per US dollar) was estimated as $\hat{\beta}_p = -0.47$. This yields $|\widehat{\mathrm{CV}}|/N = -\hat{\lambda}/\hat{\beta}_p = 8.149$, meaning that the per-person estimate of the benefit gained from the beach is about 8 dollars per year.

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A Exercises

(a) Consider an individual whose preference is represented by a utility function

$$U(x, y) := x^{\alpha} y^{1-\alpha}. \tag{A.1}$$

The budget constraint for this individual is given by

$$px + y = m. \tag{A.2}$$

Please answer the following questions.

i) Derive the demand functions $x^d(p, m)$ and $y^d(p, m)$.

ii) Compute the income elasticity of $x^d(p, m)$.

iii) Derive the indirect utility function $V(p, m)$.

iv) Denoting by u a fixed level of utility, derive the compensated demand functions $x^c(p, u)$ and $y^c(p, u)$.

v) Derive the expenditure function $e(p, u)$.

vi) Using this particular example:

1) Please verify that the statements in Theorem 3.1 are all correct;

2) Similarly, verify that the statement in Theorem 3.2 is correct.

vii) Let $\bar{m}$ be the initial level of income. Suppose that the price p of x declines from $\bar{p}$ to $\bar{p}' < \bar{p}$.

1) Compute CV and EV associated with the price change by using their definitions.

2) Verify that the statement in Theorem 3.3 is correct.

3) Verify that the statement in Theorem 3.4 is correct.

4) Compute ΔCS based on the definition (3.25).

5) Verify that the statement in Theorem 3.5 is correct.

(b) Consider an individual whose preference is represented by a utility function

$$U(x, y, q) := qx^{\alpha} + y^{\alpha}. \tag{A.3}$$

The budget constraint for this individual is given by

$$px + y = m. \tag{A.4}$$

Please answer the following questions.

i) Derive the demand functions $x^d(p, q, m)$ and $y^d(p, q, m)$.

ii) Derive the indirect utility function $V(p, q, m)$.

iii) Denoting by u a fixed level of utility, derive the compensated demand functions $x^c(p, q, u)$ and $y^c(p, q, u)$.

iv) Derive the expenditure function $e(p, q, u)$.

v) Using this particular example:

1) Please verify that the statements in Theorem 4.1 are all correct;

2) Similarly, verify that the statement in Theorem 4.2 is correct;

3) Also, verify that the statement in Theorem 4.4 is correct.

vi) Let $\bar{m}$ be the initial level of income. Suppose that q increases from $\bar{q}$ to $\bar{q}' > \bar{q}$.

1) Compute the associated CV and EV by using their definitions.

2) Is x a weak complement to q?

3) Is x a non-essential good?

4) Verify that the statement in Lemma 4.1 is correct.

5) Verify that the statement in Theorem 4.5 is correct.

(c) Consider the household production model

$$\tilde{U}(w, y) := w^{\alpha} y^{1-\alpha}, \tag{A.5}$$

where w is produced by the household production function

$$w = f(x, q) := qx. \tag{A.6}$$

The budget constraint for this individual is given by

$$px + y = m. \tag{A.7}$$

Please answer the following questions.

i) Derive the demand functions $x^d(p, q, m)$ and $y^d(p, q, m)$.

ii) Derive the indirect utility function $V(p, q, m)$.

iii) Denoting by u a fixed level of utility, derive the compensated demand functions $x^c(p, q, u)$ and $y^c(p, q, u)$.

iv) Derive the expenditure function $e(p, q, u)$.

v) Let $\bar{m}$ be the initial level of income. Suppose that q increases from $\bar{q}$ to $\bar{q}' > \bar{q}$.

1) Compute the associated CV and EV by using their definitions.

2) Verify that the statement in Theorem 4.6 is correct.